POINT SINGULARITIES ON 2D SURFACES

by

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Abstract

This work studies point vortices on a sphere and complex point singularities on a plane. The motivation for the study is to get deeper understanding of the dynamics of symmetric configurations of point vortices and point singularities. Equations of point vortex motion are derived from the Euler equations. Geometric description of the phase space is given along with symplectic structure and Lie-Poisson brackets. Symplectic reduction is performed and reduced Hamiltonian is found. Configuration matrix approach is used to find fixed equilibrium configurations of point singularities and relative equilibrium configurations of point vortices. Based on this method, relative equilibria in the form of tetrahedron, octahedron, cube, icosahedron, dodecahedron are described. Using energy-momentum method conducted study of stability of general tetrahedral and octahedral configurations. For the cubic, icosahedral and dodecahedral cases studied stability of superpositions of axis-symmetric configurations. For the tetrahedral, cubic and icosahedral configuration regions of stability are plotted. Instability results for special cases of cubic and icosahedral configurations are proved.

Fixed equilibrium configurations of point singularities on a plane are found. Theorems about existence and uniqueness of the equilibria are proved. For each of the configuration, singular value decomposition is performed. The singular values are used to obtain probability distribution and Shannon entropy for the configurations is computed. Relative equilibria for even and odd number of point singularities are described.

Relative equilibria for 2, 3, 4 point singularities are studied. For higher number of singularities method of finding relative equilibria is provided.

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Preface

Study of point singularities on 2D surfaces is motivated by several models in physics. The first and the most famous is a model point vortices on a plane. It is an old model and even though it has been studied for more then a century, it still has a lot of open questions. The model has found its applications in different fields of theoretical physics, starting from fluid dynamics and ending in quantum mechanics[1]. Another related model is a model of spiral vortices and sources/sinks on a plane. This is just an extension of the point vortex model. One more extension of the point vortex model, is a model of point vortices on a sphere. These three models can be studied in one framework, which I am going to do in this work.

The field of vortex dynamics has started in the prominent work of H. Helmholtz [Hel58]. In this work, Helmholtz introduced his laws of vorticity evolution and described first principles of vortex evolution. These results have already became a classic results and can be found in every textbook on fluid dynamics.[2] Starting from Helmholtz, vorticity field description became a powerful tool in the theory of continuous vector fields.

The motion of straight, parallel, infinitely thin vortex filaments (rectilinear vortices) in incompressible inviscid fluid was one of the fields of research started by Helmholtz. In one of the first published lectures on vortex dynamics[Kir77], Kirchhoff showed that

[1]Nice introduction to the applications of point vortices can be found in [Lug83].

[2]See for example Kirchhoff [Kir77] , Lamb[Lam32], Prandtl[Pra52], Milne-Thomson[MT55], Batchelor[Bat67], etc.

these filaments can be identified with points on a plane (point vortices) and their motion is governed by Hamilton's equations.

Model of point vortices on a sphere is a closely model. It describes motion of thin vortex filaments in a thin spherical layer of ideal incompressible fluid. Motivation for the study in this field comes from the atmospherical sciences. Clearly, model of point vortices on sphere gives more accurate description of geoatmospheric phenomena compared to the planar vortex model. First works in this direction date back to E. Zermello[Zer02], where he defined point vortex on a sphere and derived the equations of motion. Recent interest was stired up by the work of Bogomolov[Bog77]. In his article he rederived the equations of motion and showed their Hamiltonian structure.

As it was shown by Yudovich[Yud63], 2D Euler equations (in a p with bounded initial vorticity always have unique solutions. Keeping this result in mind, Marchioro and Pulvirenti[MP94] proved that point vortices are weak solutions of planar Euler equations and justified the model by proving that regions of localized vorticity stay localized up to a certain time T and their centers move according to the point vortex equations. Similar result for the point vortices on a sphere was obtained by Garra[R.13].

Integrability of the point vortex equations with small number of vortices was first proven by Gröbli[Gro77]. He proved that for three point vortices the system has 3 integrals of motion in convolution and thus the system is integrable. Later Synge[Syn49], Novikov[Nov76] and Aref[Are79] independently proved similar results. Proof of integrability of three point vortex problem on a sphere was given independently by P. K. Newton and R. Kidambi in [KN98] and by A. V. Borisov and V. G. Lebedev in [BL98].

A curious observation of floating magnets served as one of the motivations for the study of stability of relative equilibrium configurations of point vortices on a plane. Lord Kelvin in his work[Tho78] pointed out similarity between point vortices on a plane and floating magnets. After this work a lot of attention have been brought to the problem of stability of planar configurations of point vortices. Lord Kelvin showed that regular vortex polygons (regular polygons with point vortices of equal strength in its vertices) are stable relative equilibrium configurations for $N < 7$ and unstable for $N > 7$. The question of stability for $N = 7$ remained open for more then a century and was answered in the recent work by Kurakin and Yudovich[KY02], where they showed that for $N = 7$ regular vortex polygon is stable. Even though problem of finding equilibrium configurations and their stability analysis have been extensively studied there are still many unanswered questions.

Relative equilibrium configurations of point vortices on a sphere can be described using elegant linear algebra approach as it was show by Jamaloodean and Newton[JN06]. They showed that any given geometric configuration of point vortices on a sphere has a corresponding configuration matrix. This matrix has nontrivial null space if and only if the corresponding vortex configuration is in relative equilibrium. They also showed that all the Archimedean solids are relative equilibrium configurations. Further developments of this work is done by myself and Newton[MON10]. We studied wider class of symmetric configurations represented by Platonic solids and showed that among all of them, only cuboctahedron and icosidodecahedron are relative equilibrium configurations. Another description of relative equilibrium configurations can be found in [LMR01]. There authors enumerated all the discrete subgroups of $\mathrm{SO}(3)$ to classify all the possible symmetric equilibrium configurations with equal intensities of point vortices.

There are not many results on the stability of equilibrium configurations of point vortices on a sphere. The main object of interest in most of the works are configurations of vortex rings on a sphere. In [CMS03] authors study stability of longitudinal ring of equal vortices with additional vortex at the pole on the sphere. They showed regions of stability and conducted bifurcation analysis for small numbers of N. In two separate works [Hal80, LP05] authors studied stability of vortex streets on the sphere. Stability of Platonic solids point vortex configurations have been conducted in [Kur04]. Stability of relative equilibrium configurations based on Archimedean solids have been studied in [MON10]. All of these woks considered vortex configurations with equal or possible two different strengths. These configurations represent spherical equivalent of planar regular polygon configurations[3]. More comprehensive approach is used in [PM98], where authors conducted stability analysis of 3 point vortex relative equilibrium configurations.

Another direction, which is closely related to the problem of point vortices, is dynamics of complex singularities on 2D surfaces. These singularities describe motion of the spiral vortices in the ideal fluid. Equations of motion for these type of singularities in a plane are similar to the point vortex equations. The only difference is the intensities, which are not real, but complex numbers. Verification of the model can be done using hollow vortex model[SC12, CSF12, CG11]. Detailed historic review of the derivation of point singularity equations can be found in a recent work by S. G. L. Smith[Smi11].

The work contains two parts. The first part is based on author's publications [NO12a, MON10]. In it we will present the background on the point vortex model, configuration matrix approach[KN98], singular value decomposition[GVL96], Shannon entropy[NS09] and energy-momentum method[SLM91]. Then we use these methods to study relative equilibrium configurations. The first chapter introduces the methods and gives geometric description. The second deals with relative equilibria. And in the third chapter we study stability.

The second part is based on authors publications [NO12b, OVV13]. There we use the same techniques as we used in the first part, but we apply them to the different problems.The first chapter introduces the equations of motion and then we investigate its

[3]Nice overview of point vortex configurations on a plane can be found in [ANS$^+$02].

symmetries. In the second chapter of the second part we find fixed equilibria using configuration matrix approach, SVD and Shannon entropy. In the last chapter we investigate stability of the fixed equilibria.

Part I

Point vortices on a sphere

Chapter 1

Introduction

1.1 Equations of motion on 2D surface

Let us begin with derivation of equations of motion of N point vortices. We will start the derivation from the general model of ideal incompressible invicid fluid and then introduce notion of point vortex. We then show that the N point vortices represent a weak solution of 2D Euler equations. To demonstrate the details of the derivation and to be able to derive several forms of equations of motion, we will use general curvilinear coordinates.

If $\mathbf{u}$ is a velocity field of the fluid then corresponding equations of the fluid motion are Euler equations and have a following form

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = 0, \\ \nabla \cdot \mathbf{u} = 0. \end{cases}$$

By taking curl we get vorticity form of the Euler equations

$$\begin{cases} \frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}, \\ \boldsymbol{\omega} = \nabla \times \mathbf{u}, \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \tag{1.1.1}$$

where $\boldsymbol{\omega}$ is a corresponding vorticity field. In curvilinear coordinates (q_1, q_2, q_3) these equations will transform to[1]

$$\left\{\begin{aligned} \frac{\partial \omega_i}{\partial t} + \sum_{k=1}^{3}\left[\frac{u_k}{H_k}\frac{\partial \omega_i}{\partial q_k} + \frac{\omega_k}{H_i H_k}\left(u_i \frac{\partial H_i}{\partial q_k} - u_k \frac{\partial H_k}{\partial q_i}\right)\right] &= \\ = \sum_{k=1}^{3}\left[\frac{\omega_k}{H_k}\frac{\partial u_i}{\partial q_k} + \frac{u_k}{H_i H_k}\left(\omega_i \frac{\partial H_i}{\partial q_k} - \omega_k \frac{\partial H_k}{\partial q_i}\right)\right]&, \\ \omega_1 = \frac{1}{H_2 H_3}\left(\frac{\partial(u_3 H_3)}{\partial q_2} - \frac{\partial(u_2 H_2)}{\partial q_3}\right)&, \\ \omega_2 = \frac{1}{H_1 H_3}\left(\frac{\partial(u_1 H_1)}{\partial q_3} - \frac{\partial(u_3 H_3)}{\partial q_1}\right)&, \\ \omega_3 = \frac{1}{H_1 H_2}\left(\frac{\partial(u_2 H_2)}{\partial q_1} - \frac{\partial(u_1 H_1)}{\partial q_2}\right)&, \\ \frac{1}{H_1 H_2 H_3}\left(\frac{\partial(u_1 H_2 H_3)}{\partial q_1} + \frac{\partial(u_2 H_1 H_3)}{\partial q_2} + \frac{\partial(u_3 H_1 H_2)}{\partial q_3}\right) &= 0, \end{aligned}\right. \tag{1.1.2}$$

where H_1, H_2, H_3 are Lame's coefficients.

In many practical applications it is reasonable to consider the fluid motion to be 2D motion. For example in geophysics, because the height of atmosphere is small compared to the radius of the planet, atmospheric motion can be regarded as motion of ideal fluid in an infinitely thin spherical shell.

Typically, to derive 2D equations of fluid motion in a thin shell, one assumes that vector fields are independent of one of the coordinates. For the velocity field this

[1]Detailed derivation of these equations can be found in [KKR64].

means that we assume $\mathbf{u}(q_1, q_2, q_3) = \mathbf{u}(q_1, q_2)$ and then from (1.1.2) we will get that $(\omega_1, \omega_2, \omega_3) = (0, 0, \omega)$ and

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + \dfrac{u_1}{H_1}\dfrac{\partial \omega}{\partial q_1} + \dfrac{u_2}{H_2}\dfrac{\partial \omega}{\partial q_2} = 0, \\ \omega = \dfrac{1}{H_1 H_2}\left(\dfrac{\partial (u_2 H_2)}{\partial q_1} - \dfrac{\partial (u_1 H_1)}{\partial q_2}\right), \\ \dfrac{1}{H_1 H_2}\left(\dfrac{\partial (u_1 H_2)}{\partial q_1} + \dfrac{\partial (u_2 H_1)}{\partial q_2}\right) = 0, \end{cases} \tag{1.1.3}$$

From the third equation in (1.1.3) we can see that the velocity field can be described using stream function ψ such that $(u_1, u_2) = \nabla^{\perp}\psi$, or

$$\begin{cases} u_1 = \dfrac{1}{H_2}\dfrac{\partial \psi}{\partial q_2}, \\ u_2 = -\dfrac{1}{H_1}\dfrac{\partial \psi}{\partial q_1}. \end{cases} \tag{1.1.4}$$

With this substitution equations (1.1.3) will transform to

$$\begin{cases} \omega = -\dfrac{1}{H_1 H_2}\left[\dfrac{\partial}{\partial q_1}\left(\dfrac{H_2}{H_1}\dfrac{\partial \psi}{\partial q_1}\right) + \dfrac{\partial}{\partial q_2}\left(\dfrac{H_1}{H_2}\dfrac{\partial \psi}{\partial q_2}\right)\right], \\ \dfrac{1}{H_1 H_2}\left(\dfrac{\partial \omega}{\partial t} + \dfrac{\partial \psi}{\partial q_2}\dfrac{\partial \omega}{\partial q_1} - \dfrac{\partial \psi}{\partial q_1}\dfrac{\partial \omega}{\partial q_2}\right) = 0, \end{cases} \tag{1.1.5}$$

Motion of thin vortex filaments oriented orthogonally to the 2D surface can be described by the model of a point vortex[2].

[2]Proof of the localization property and validation of the model local in time for the planar case can be found in Marchioro's book[MP94]. Proof of the same property for the sphere can be found in recent work by Garra[R.13].

Definition 1.1.1. Point vortex is a δ-function singularity of 2D vorticity field.

In planar case a point vortex produces vorticity field

$$\omega(x,y) = \Gamma\delta(x-x_0)\delta(y-y_0), \tag{1.1.6}$$

where Γ is called intensity(circulation) of point vortex and (x_0, y_0) is a position of point vortex.

On smooth compact surfaces, in order for Kelvin circulation theorem to hold, vorticity field corresponding to a point vortex should be

$$\omega(q_1,q_2) = \frac{\Gamma}{H_1H_2}\delta(q_1-q_1^{(0)})\delta(q_2-q_2^{(0)}) - \frac{\Gamma}{C}, \tag{1.1.7}$$

where the last term helps to preserve total vorticity and C is a circulation of uniform vorticity field around infinitely small unit circle. For example, in spherical coordinates (r,θ,ϕ) on a unit sphere a point vortex produces the vorticity field

$$\omega(\theta,\phi) = \frac{\Gamma}{\sin\theta}\delta(\theta-\theta_0)\delta(\phi-\phi_0) - \frac{\Gamma}{4\pi}, \tag{1.1.8}$$

Linear superposition of point vortices will create following vorticity field

$$\omega^N(q_1,q_2) = \sum_{i=1}^{N}\frac{\Gamma_i}{H_1H_2}\delta(q_1-q_1^{(i)})\delta(q_2-q_2^{(i)}) - s\frac{\Gamma}{C}, \tag{1.1.9}$$

where $\Gamma = \sum_{i=1}^{N}\Gamma_i$ and $s=0$ in planar case, $s=1$ on compact surface.

Proposition 1.1.2. [3] *Linear superposition of N independent point vortices is a weak solution of 2D vorticity equations. Positions of point vortices change according to the equations*

$$\begin{cases} \dfrac{dq_1^{(i)}}{dt} = -\dfrac{\partial \psi^N(q_1^{(i)}, q_2^{(i)})}{\partial q_1}, \\ \dfrac{dq_1^{(i)}}{dt} = \dfrac{\partial \psi^N(q_1^{(i)}, q_2^{(i)})}{\partial q_2}, \end{cases} \tag{1.1.10}$$

where $\psi^N = \omega^N * \hat{\psi}$ and $\hat{\psi}$ is a fundamental solution of Laplace-Beltrami operator, i.e. it is the solution of the equation

$$-\frac{1}{H_1H_2}\left[\frac{\partial}{\partial q_1}\left(\frac{H_2}{H_1}\frac{\partial\hat{\psi}}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{H_1}{H_2}\frac{\partial\hat{\psi}}{\partial q_2}\right)\right] = \frac{1}{H_1H_2}\delta(q_1 - q_1^{(i)})\delta(q_2 - q_2^{(i)}) - s\frac{1}{C}. \tag{1.1.11}$$

Proof. To show that ω^N is a weak solution of 2D vorticity equation, we will plug in ω^N and ψ^N into the second equation in (1.1.3), multiply it by a test function $\phi \in C_c^2(D)$ and integrate over D. After these manipulations we will get

$$\int_D \left[\sum_{i=1}^N \frac{\Gamma_i}{H_1H_2}\frac{dq_1^{(1)}}{dt}\delta'(q_1 - q_1^{(i)})\delta(q_2 - q_2^{(i)}) + \frac{\Gamma_i}{H_1H_2}\frac{dq_2^{(1)}}{dt}\delta(q_1 - q_1^{(i)})\delta'(q_2 - q_2^{(i)}) + \right.$$
$$+\frac{\partial\psi^N}{\partial q_2}\left(\frac{\Gamma_i}{H_1H_2}\delta'(q_1 - q_1^{(i)})\delta(q_2 - q_2^{(i)}) + \delta(q_1 - q_1^{(i)})\delta(q_2 - q_2^{(i)})\frac{\partial}{\partial q_1}\frac{\Gamma_i}{H_1H_2}\right) -$$
$$\left. -\frac{\partial\psi^N}{\partial q_1}\left(\frac{\Gamma_i}{H_1H_2}\delta(q_1 - q_1^{(i)})\delta'(q_2 - q_2^{(i)}) + \delta(q_1 - q_1^{(i)})\delta(q_2 - q_2^{(i)})\frac{\partial}{\partial q_2}\frac{\Gamma_i}{H_1H_2}\right)\right]\phi dS =$$
$$= 0,$$

[3] Proof of this proposition for the planar case can be found in [MP94]. Different derivation of equations of motion on compact surfaces can be found in [Zer02, Kim99, Bog77].

For any $A(q)$ and compactly supported $\phi(q)$ we have

$$\begin{aligned}\int_D A(q)\delta'(q-q_0)\phi(q)dq = -\int_D \delta(q-q_0)(A(q)\phi(q))'dq = \\ = -\int_D \delta(q-q_0)(A'(q)\phi(q)+A(q)\phi'(q))dq = \\ = -A'(q_0)\phi(q_0) - A(q_0)\phi'(q_0) = \\ = -A'(q_0)\int_D \delta(q-q_0)\phi(q)dq - A(q_0)\int_D \delta(q-q_0)\phi'(q)dq = \\ = -A'(q_0)\int_D \delta(q-q_0)\phi(q)dq + A(q_0)\int_D \delta'(q-q_0)\phi(q)dq = \\ = -\int_D A'(q_0)\delta(q-q_0)\phi(q)dq + \int_D A(q_0)\delta'(q-q_0)\phi(q)dq,\end{aligned}$$

and

$$\int_D A(q)\delta(q-q_0)\phi(q)dq = A(q_0)\phi(q_0) = \int_D A(q_0)\delta(q-q_0)\phi(q)dq.$$

Using these observations and after some simplifications we get

$$\int_D \sum_{i=1}^{N} \left[\left.\frac{\Gamma_i}{H_1H_2}\right|_{(q_1^{(i)},q_2^{(i)})} \left(\frac{dq_1^{(1)}}{dt} + \frac{\partial \psi^N(q_1^{(i)},q_2^{(i)})}{\partial q_2} \right) \delta'(q_1-q_1^{(i)})\delta(q_2-q_2^{(i)}) + \right.$$
$$\left. + \left.\frac{\Gamma_i}{H_1H_2}\right|_{(q_1^{(i)},q_2^{(i)})} \left(\frac{dq_2^{(1)}}{dt} - \frac{\partial \psi^N(q_1^{(i)},q_2^{(i)})}{\partial q_1} \right) \delta(q_1-q_1^{(i)})\delta'(q_2-q_2^{(i)}) \right] \phi dS = 0, \tag{1.1.12}$$

Since $\delta'(q_1-q_1^{(i)})\delta(q_2-q_2^{(i)})$, $\delta(q_1-q_1^{(i)})\delta'(q_2-q_2^{(i)})$, $i=1,...,N$ are linearly independent, from (1.1.12) we get that point vortices are week solutions of Euler equations. □

1.2 Equations of motion on a sphere

Since in this work we will be studying motion of point vortices on a sphere, let us derive different forms of the equations of motion of point vortices on a unit sphere. The most used coordinates on a sphere are the regular spherical coordinates. Let

$$q_1 = \theta, \ -\pi < \theta \leq \pi,$$
$$q_2 = \phi, \ 0 \leq \phi < 2\pi,$$
$$q_3 = r = 1.$$

Then $H_1 = 1, H_2 = \sin\theta$ and $\hat{\psi}$ is

$$\hat{\psi}(\theta, \phi) = -\frac{1}{4\pi} \ln\left[1 - \cos(\gamma_i)\right], \tag{1.2.1}$$

where $\gamma_i = \gamma_i(\theta, \phi)$ is the central angle between a point vortex (θ_i, ϕ_i) and a point (θ, ϕ) and

$$\cos\gamma_i = \cos\theta\cos\theta_i + \sin\theta\sin\theta_i\cos(\phi - \phi_i).$$

Then the equations of motion (1.1.10) became

$$\sin\theta_i\dot{\phi}_i = \frac{1}{4\pi}\sum_{j=1, i\neq j}^{N} \Gamma_j \frac{\kappa_{ij}}{1 - \cos\gamma_{ij}},$$
$$\dot{\theta}_i = -\frac{1}{4\pi}\sum_{j=1, i\neq j}^{N} \Gamma_j \frac{\sin\theta_j\sin(\phi_i - \phi_j)}{1 - \cos\gamma_{ij}},$$

where $\gamma_{ij} = \gamma_j(\phi_i, \theta_i)$ and $\kappa_{ij} = \sin\theta_i\cos\theta_j - \cos\theta_i\sin\theta_j\cos\left(\phi_i - \phi_j\right)$.

If we embed the unit sphere into $\mathbb{R}^3$ (use cartesian coordinates and place the center of the sphere into the origin) the equations (1.2.4) will transform to

$$\begin{cases} \dot{\mathbf{x}}_i = \sum_{j=1, j\neq i}^{N} \frac{\Gamma_j}{2\pi} \frac{\mathbf{x}_j \times \mathbf{x}_i}{(\mathbf{x}_i - \mathbf{x}_j)^2}, \\ ||\mathbf{x}_i|| = 1, \ i = 1, \ldots, N. \end{cases} \tag{1.2.2}$$

where $\mathbf{x}_i = (x_i, y_i, z_i)$ coordinates of a i-th point vortex.

As it is shown in [New01], equations (1.2.2) can be reduced to equations for $l_{ij} = ||\mathbf{x}_i - \mathbf{x}_j||, 1 \leq i < j \leq N$. To get these equations, subtract corresponding equations for ith and jth vortex and dot multiply the difference by $(\mathbf{x}_i - \mathbf{x}_j)$. After some simplifications we will get

$$(\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j) \cdot (\mathbf{x}_i - \mathbf{x}_j) =$$
$$= \sum_{k=1, k\neq i, k\neq j}^{N} \frac{\Gamma_k}{2\pi} \left[\frac{\mathbf{x}_i \cdot \mathbf{x}_j \times \mathbf{x}_k}{(\mathbf{x}_j - \mathbf{x}_k)^2} - \frac{\mathbf{x}_i \cdot \mathbf{x}_j \times \mathbf{x}_k}{(\mathbf{x}_i - \mathbf{x}_k)^2} \right],$$

Since $\frac{dl_{ij}^2}{dt} = 2(\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j) \cdot (\mathbf{x}_i - \mathbf{x}_j)$, then

$$\frac{dl_{ij}^2}{dt} = \sum_{k=1, k\neq i, k\neq j}^{N} \frac{\Gamma_k V_{ijk}}{\pi} \left(\frac{1}{l_{jk}^2} - \frac{1}{l_{ik}^2} \right), \tag{1.2.3}$$

where $V_{ijk} = \mathbf{x}_i \cdot \mathbf{x}_j \times \mathbf{x}_k$. These equations depend only on l_{ij}^2 and describe relative motion of point vortices on a sphere.

Another useful form of equations of motions can be obtained if we introduce cylindrical coordinates on a sphere. Let

$$q_1 = z, \ -1 \leq z \leq 1,$$

$$q_2 = \phi, \ 0 \leq \phi < 2\pi,$$

$$q_3 = r = 1.$$

Then $H_1 = 1, H_2 = \sqrt{1-z^2}$ and equations of motion will be

$$\sin\theta_i \dot{\phi}_i = \frac{1}{4\pi}\sum_{j=1, i\neq j}^{N} \Gamma_j \frac{m_{ij}}{1-k_{ij}},$$
$$\dot{\theta}_i = -\frac{1}{4\pi}\sum_{j=1, i\neq j}^{N} \Gamma_j \frac{\sqrt{1-z_j^2}\sin(\phi_i-\phi_j)}{1-k_{ij}},$$

where $k_{ij} = z_i z_j + \sqrt{1-z_i^2}\sqrt{1-z_j^2}\cos(\phi_i - \phi_j)$ and $m_{ij} = \sqrt{1-z_i^2}z_j - z_i\sqrt{1-z_j^2}\cos(\phi_i - \phi_j)$.

1.3 Hamiltonian structure and integrals of motion

As it is shown in [Bog77, Zer02, Kim99, New01] equations of point vortex motion on a sphere can be written in a Hamiltonian form. Canonical variables and Hamiltonian function for the system are

$$H = \frac{1}{4\pi}\sum_{i<j}\Gamma_i\Gamma_j \ln l_{ij},$$
$$p_i = \sqrt{|\Gamma_i|}\cos\theta_i, \quad q_i = \mathrm{sign}(\Gamma_i)\sqrt{|\Gamma_i|}\phi_i,$$
$$\dot{p}_i = \frac{\partial H}{\partial q_i}, \quad \dot{q}_i = -\frac{\partial H}{\partial p_i}. \tag{1.3.1}$$

Since the Hamiltonian is invariant under the action of $\mathrm{SO}(3)$, by Noether's theorem we will have 3 ($\dim \mathrm{SO}(3) = 3$) conserved quantities

$$\begin{aligned} c_1 &= \sum_{i=1}^{N} \Gamma_i x_i = \sum_{i=1}^{N} \Gamma_i \sin\theta_i \cos\phi_i = \sum_{i=1}^{N} \Gamma_i \sqrt{1 - z_i^2} \cos\phi_i, \\ c_2 &= \sum_{i=1}^{N} \Gamma_i x_i = \sum_{i=1}^{N} \Gamma_i \sin\theta_i \sin\phi_i = \sum_{i=1}^{N} \Gamma_i \sqrt{1 - z_i^2} \sin\phi_i, \\ c_3 &= \sum_{i=1}^{N} \Gamma_i x_i = \sum_{i=1}^{N} \Gamma_i \cos\theta_i = \sum_{i=1}^{N} \Gamma_i z_i. \end{aligned} \tag{1.3.2}$$

Vector $\mathbf{c} = (c_1, c_2, c_2)$ is called center of vorticity and $\mathbf{M} = \mathbf{c}/(\sum_{i=1}^{N})\Gamma_i$ is a moment of vorticity. To prove that the components of vector $\mathbf{c}$ are integrals of motion, one should consider $\dot{\mathbf{c}} = \sum_{i=1}^{N} \Gamma_i \dot{\mathbf{x}}_i$ and use (1.2.2) along with skew-symmetry of vector product to show that $\dot{\mathbf{c}} = \mathbf{0}$.

1.4 Geometric description

Dynamical system of N point vortices on a sphere admits nice geometric description. Since point vortices can not collide the phase space of the system can be written as

$$P = \{\mathbf{s} = (s_1, \ldots, s_N) \in S^2 \times \ldots \times S^2 | s_i \neq s_j, i \neq j, 1 \leq i, j \leq N\}. \tag{1.4.1}$$

The symplectic structure on the phase space is given by

$$\tilde{\omega} = \sum_{i=0}^{N} \frac{1}{\Gamma_i} \pi_i^* \omega_{S^2} \tag{1.4.2}$$

where Γ_i are intensities of point vortices, π_i - projection onto the i-th copy of S^2 and ω_{S^2} is the area form on S^2.

Corresponding Poisson bracket for $f, g \in C(P)$ can be written as

$$\{f, g\} = \tilde{\omega}(\mathbf{X}_f, \mathbf{X}_g). \tag{1.4.3}$$

As it was shown in previous section, the system is a Hamiltonian system with Hamiltonian $H = -\sum_{i=1}^{N} \Gamma_i \Gamma_j \ln l_{ij}$, where l_{ij} is an euclidean distance between vortices i and j.

After embedding S^2 into $\mathbb{R}^3$ we can rewrite the phase space, form $\tilde{\omega}$ and Poisson bracket as

$$P = \{\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N) \in (\mathbb{R}^3)^N | \, ||\mathbf{x}_i|| = 1, ||\mathbf{x}_j|| = 1, \mathbf{x}_i \neq \mathbf{x}_j, i \neq j, 1 \leq i, j \leq N\},$$

$$\tilde{\omega} = -\sum_{i=1}^{N} \frac{1}{\Gamma_i} \left(x_i dy_i \wedge dz_i + y_i dz_i \wedge dx_i + z_i dx_i \wedge dy_i \right), \tag{1.4.4}$$

$$\{f, g\} = -\sum_{i=1}^{N} \frac{1}{\Gamma_i} \mathbf{x}_i \cdot \left(\nabla_i f \times \nabla_i g \right), \tag{1.4.5}$$

where ∇_i is a gradient on i-th copy of $\mathbb{R}^3$. Notice, that since we have one redundant variable, form $\tilde{\omega}$ is degenerate and thus it is not symplectic any more. But if we introduce chart containing only 2 spacial coordinates, the form will became symplectic.

If we introduce spherical coordinates on each copy of S^2 then all of the above can be rewritten as

$$P = \{\mathbf{s} = (s_1, \dots, s_N) \in (S^2)^N | s_i = (\theta_i, \phi_i), s_i \neq s_j, i \neq j, 1 \leq i, j \leq N\},$$

$$\tilde{\omega} = \sum_{i=1}^{N} \frac{1}{\Gamma_i} \sin\theta_i d\phi_i \wedge d\theta_i,$$

$$\{f, g\} = \sum_{k=1}^{N} \frac{1}{\Gamma_k} \left(\frac{\partial f}{\partial \cos\theta_k} \frac{\partial g}{\partial \phi_k} - \frac{\partial f}{\partial \phi_k} \frac{\partial g}{\partial \cos\theta_k} \right),$$

$$\{\phi_i, \cos\theta_j\} = \frac{\delta_{ij}}{\Gamma_i}. \tag{1.4.6}$$

And for cylindrical coordinates on each copy of S^2 we will get

$$P = \{\mathbf{u} = (u_1, \ldots, u_N) \in (S^2)^N | u_i = (z_i, \phi_i), u_i \neq u_j, i \neq j, 1 \leq i, j \leq N\},$$
$$\tilde{\omega} = \sum_{i=1}^{N} \frac{1}{\Gamma_i} dz_i \wedge d\phi_i,$$
$$\{f, g\} = \sum_{k=1}^{N} \frac{1}{\Gamma_k} \left(\frac{\partial f}{\partial z_k} \frac{\partial g}{\partial \phi_k} - \frac{\partial f}{\partial \phi_k} \frac{\partial g}{\partial z_k} \right),$$
$$\{\phi_i, z_j\} = \frac{\delta_{ij}}{\Gamma_i}. \tag{1.4.7}$$

Since the diagonal action $\mathrm{SO}(3)$ on P is canonical, the momentum map[4] will have 3 components which are proportional to the coordinates of the center of vorticity.

Proposition 1.4.1. *Momentum map for the system of N point vortices on a sphere embedded in $\mathbb{R}^{3N}$ is*

$$\mathbf{J}(\mathbf{x}) = -\mathbf{c} = -\sum_{i=1}^{N} \Gamma_i \mathbf{x}_i. \tag{1.4.8}$$

Proof. By the definition $\mathbf{J} : P \to TP^*$ is a momentum map if for any $F \in \mathcal{F}(P)$, $\mathbf{x} \in \mathbb{R}^{3N}$, $\boldsymbol{\xi} = (\xi, \xi, \ldots, \xi) \in (\mathrm{so}(3))^N = TP$ we have

$$\{F, < \mathbf{J}(\mathbf{x}), \boldsymbol{\xi} >\} = \xi_P[F], \tag{1.4.9}$$

where $\xi_P[F]$ is a Lie derivative of F and

$$< \mathbf{J}(\mathbf{x}), \boldsymbol{\xi} >= \sum_{l=1}^{N} < \mathbf{J}_l(\mathbf{x}), \xi >_d = \sum_{l=1}^{N} \mathbf{J}_l(\mathbf{x}) \cdot \xi, \tag{1.4.10}$$

[4]For details about momentum mappings see the section 3 and [AM78].

where $\mathbf{J}_l = \pi_l(\mathbf{J})$ - projection of $\mathbf{J}$ onto l-th copy of $\mathbb{R}^3$, $< \cdot, \cdot >_d$ - duality relation beween $\mathrm{so}(3)$ and $\mathrm{so}(3)^*$ (in case if these spaces are embedded into $\mathbb{R}^3$, the relation is a dot product).

From the properties of dot product we have

$$\begin{aligned}\{F, < \mathbf{J}(\mathbf{x}), \xi >\} &= -\sum_{i=1}^{N} \frac{1}{\Gamma_i} \mathbf{x}_i \cdot (\nabla_i F \times \nabla_i < \mathbf{J}(\mathbf{x}), \boldsymbol{\xi} >) = \\ &= -\sum_{i=1}^{N} \frac{1}{\Gamma_i} \nabla_i F \cdot (\nabla_i < \mathbf{J}(\mathbf{x}), \boldsymbol{\xi} > \times \mathbf{x}_i) = \\ &= \sum_{i=1}^{N} \nabla_i F \cdot (\nabla_i (\sum_{l=1}^{N} \frac{-\mathbf{J}_l(\mathbf{x})}{\Gamma_i} \cdot \xi) \times \mathbf{x}_i) = \\ &= \sum_{i=1}^{N} \nabla_i F \cdot (\xi \times \mathbf{x}_i) = \xi_P[F],\end{aligned}$$

Since the relations hold for any F, $\mathbf{x}_i$ and ξ, thus

$$\nabla_i (\sum_{l=1}^{N} \frac{-\mathbf{J}_l(\mathbf{x})}{\Gamma_i} \cdot \xi) = \xi, \quad i = 1, .., N. \tag{1.4.11}$$

In cartesian coordinates

$$\begin{aligned}\frac{-(\mathbf{J}_l(\mathbf{x}))'_{3i+1}}{\Gamma_i} \xi_x &= \xi_x, \\ \frac{-(\mathbf{J}_l(\mathbf{x}))'_{3i+2}}{\Gamma_i} \xi_y &= \xi_y, \\ \frac{-(\mathbf{J}_l(\mathbf{x}))'_{3i+3}}{\Gamma_i} \xi_z &= \xi_z, \\ i &= 1, .., N.\end{aligned}$$

Thus

$$\frac{-(\mathbf{J}_l(\mathbf{x}))'_{3l+1}}{\Gamma_l} = \frac{-(\mathbf{J}_l(\mathbf{x}))'_{3l+2}}{\Gamma_l} = \frac{-(\mathbf{J}_l(\mathbf{x}))'_{3l+3}}{\Gamma_l} = 1, \Rightarrow$$

$$\mathbf{J}_l(\mathbf{x}) = -\Gamma_l \mathbf{x}_l,$$

which proves the proposition. □

Now, since for any $F \in \mathcal{F}(P)$, $\mathbf{x} \in \mathbb{R}^{3N}$ and $\boldsymbol{\xi} \in TP^*$

$$\{F, < \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(\mathbf{x}), \boldsymbol{\xi} >\} = \{F, < \mathbf{J}(g(\mathbf{x})), \boldsymbol{\xi} >\}$$

where $\mathrm{Ad}^*_g : TP^* \to TP^*$ is a lifted coadjoint action[5] of $g \in \mathrm{SO}(3)$ on TP^* (in our case it is a rotation of vector $\mathbf{J}$ by g). Thus

$$\mathrm{Ad}^*_{g^{-1}} \mathbf{J}(\mathbf{x}) = \mathbf{J}(g(\mathbf{x})).$$

So $\mathbf{J}$ is equivariant under the adjoint action of $g \in \mathrm{SO}(3)$ on TP^*. From the equivariance or from

$$||\mathbf{J}||^2 = \left(\sum_{i=1}^{N} \Gamma_i\right)^2 - \sum_{i<j} \Gamma_i \Gamma_j l_{ij}^2, \tag{1.4.12}$$

we have that $||\mathbf{J}||^2$ is invariant under coadjoint action of $\mathrm{SO}(3)$ and thus any smooth function of $||\mathbf{J}||^2$ is also invariant. Since the components of $\mathbf{J}$ are constant of motion, any smooth function of the components and $||\mathbf{J}||^2$ will be a Casimir function. And also, since the group $\mathrm{SO}(3)$ is compact and its action is proper on both P and TP^*, level sets of $||\mathbf{J}||^2 = \mathrm{const} \neq 0$ will define symplectic leaves of P.

1.5 Symplectic reduction

The system has a rotational symmetry and since the action of compact symmetry group $\mathrm{SO}(3)$ on P is proper, P admits reduction $W = P/\mathrm{SO}(3)$ with W being symplectic

[5] See [AM78] for the general construction of conjoint lifts.

manifold[6]. In order to study the reduced space, let us introduce special coordinates on the unreduced space. It is clear that if $\mathbf{c} \neq \mathbf{0}$ the phase space P is isomorphic to

$$P = \{\mathbf{c}, \Gamma_1 \mathbf{x}_1 - \Gamma_2 \mathbf{x}_2, \ldots, \Gamma_{N-1}\mathbf{x}_{N-1} - \Gamma_N \mathbf{x}_N\} = \mathrm{SO}(3) \times U, \tag{1.5.1}$$

where

$$U = \{\theta_1\} \times (S^2)^{N-2}. \tag{1.5.2}$$

and θ_1 is an angle between vector $\mathbf{c}$ and $\mathbf{x}_1$. The reduced space U has the dimension

$$1 + \dim(S^2) * (N-2) = 1 + 2N - 4 = 2N - 3 = \dim(P) - \dim(\mathrm{SO}(3)).$$

In order to obtain a chart for the reduced space U we use the cartesian coordinates $\mathbf{x}_i,\ i = 1, \ldots, N$. It is customary to derive the equations of motion in terms of

$$\mathbf{y}_0 = \mathbf{c}, \tag{1.5.3}$$

$$\mathbf{y}_1 = \Gamma_1 \mathbf{x}_1 - \Gamma_2 \mathbf{x}_2, \tag{1.5.4}$$

$$\ldots,$$

$$\mathbf{y}_{N-1} = \Gamma_{N-1}\mathbf{x}_{N-1} - \Gamma_N \mathbf{x}_N. \tag{1.5.5}$$

[6]See [AM78] for details about symplectic reduction.

The transformation matrix of this change of variables is

$$M_{\mathbf{x}\to\mathbf{y}} = \begin{pmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 & \dots & \Gamma_{N-1} & \Gamma_N \\ \Gamma_1 & -\Gamma_2 & 0 & \dots & 0 & 0 \\ 0 & \Gamma_2 & -\Gamma_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \Gamma_{N-1} & -\Gamma_N \end{pmatrix}, \tag{1.5.6}$$

and its inverse is

$$M_{\mathbf{y}\to\mathbf{x}} = \frac{1}{N} \begin{pmatrix} \frac{1}{\Gamma_1} & \frac{N-1}{\Gamma_1} & \frac{N-2}{\Gamma_1} & \dots & \frac{2}{\Gamma_1} & \frac{1}{\Gamma_1} \\ \frac{1}{\Gamma_1} & \frac{-1}{\Gamma_2} & \frac{N-2}{\Gamma_2} & \dots & \frac{2}{\Gamma_2} & \frac{1}{\Gamma_2} \\ \frac{1}{\Gamma_3} & \frac{-1}{\Gamma_3} & \frac{-2}{\Gamma_3} & \dots & \frac{2}{\Gamma_3} & \frac{1}{\Gamma_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\Gamma_N} & \frac{-1}{\Gamma_N} & \frac{-2}{\Gamma_N} & \dots & \frac{-N-2}{\Gamma_N} & \frac{-N-1}{\Gamma_N} \end{pmatrix}. \tag{1.5.7}$$

Thus, from (1.2.2), the equations of point vortex motion on a sphere are

$$\begin{cases} \dot{\mathbf{y}}_0 = \dot{\mathbf{c}} = \mathbf{0}, \\ \dot{\mathbf{y}}_i = \frac{\Gamma_i+\Gamma_{i+1}}{2\pi}\frac{\mathbf{x}_i\times\mathbf{x}_{i+1}}{\mathbf{y}_i^2} + \sum_{j=1, j\neq i,i+1}^{N} \frac{\Gamma_j}{2\pi}\left(\frac{\mathbf{x}_j\times\mathbf{x}_i}{(\mathbf{x}_j-\mathbf{x}_i)^2} - \frac{\mathbf{x}_j\times\mathbf{x}_{i+1}}{(\mathbf{x}_j-\mathbf{x}_{i+1})^2}\right), \\ 2\cos^{-1}(\mathbf{y}_i\cdot\mathbf{e}_z) = \cos^{-1}((\mathbf{y}_i-\mathbf{e}_z)\cdot\mathbf{e}_z), \end{cases} \tag{1.5.8}$$

where $\mathbf{e}_z = (0,0,1)$ and

$$\mathbf{x}_i = \frac{1}{\Gamma_j N}\left(\mathbf{c} - \sum_{k=1}^{i-1} k\mathbf{y}_k + \sum_{k=i}^{N-1}(N-k)\mathbf{y}_k\right),\ i = 1,\dots,N. \tag{1.5.9}$$

Now we are ready to perform the reduction. Clearly, the first equation in (1.5.8) can be reduced. This will reduce the coordinate space by 2. Additionally, one of the equations for $\mathbf{y}_i$ can be reduced and replaced by the equation for θ_k

$$\begin{aligned}
|\mathbf{c}|\frac{d\cos\theta_k}{dt} = \frac{d}{dt}(\mathbf{c}\cdot\mathbf{x}_k) = \dot{\mathbf{c}}\cdot\mathbf{x}_k + \mathbf{c}\cdot\dot{\mathbf{x}}_k = \\
= \mathbf{x}_k \cdot \sum_{i,j=1,j\neq i}^{N} \frac{\Gamma_i\Gamma_j}{2\pi}\frac{\mathbf{x}_j\times\mathbf{x}_i}{(\mathbf{x}_j-\mathbf{x}_i)^2} + (\sum_{i=1}^{N}\Gamma_i\mathbf{x}_i)\cdot \sum_{j=1,j\neq k}^{N}\frac{\Gamma_j}{2\pi}\frac{\mathbf{x}_j\times\mathbf{x}_k}{(\mathbf{x}_j-\mathbf{x}_k)^2} = \\
= \sum_{i,j=1,j\neq i\neq k}^{N}\frac{\Gamma_i\Gamma_j}{2\pi}\left(\frac{\mathbf{x}_k\cdot\mathbf{x}_j\times\mathbf{x}_i}{(\mathbf{x}_j-\mathbf{x}_i)^2} - \frac{\mathbf{x}_k\cdot\mathbf{x}_j\times\mathbf{x}_i}{(\mathbf{x}_j-\mathbf{x}_k)^2}\right) = \\
= \sum_{i,j=1,j\neq i\neq k}^{N}\frac{\Gamma_i\Gamma_j V_{kji}}{2\pi}\left(\frac{1}{l_{ji}^2}-\frac{1}{l_{jk}^2}\right).
\end{aligned} \quad (1.5.10)$$

Assume we have chosen equation for θ_1 and reduced equation for y_{N-1}. Then reconstruction equations are

$$\begin{aligned}
\mathbf{x}_1 &= R_{\mathbf{n}}(\theta_1)\hat{\mathbf{c}}, \\
\mathbf{x}_2 &= \frac{1}{\Gamma_2}(\Gamma_1\mathbf{x}_1-\mathbf{y}_1) = \frac{\Gamma_1}{\Gamma_2}R_{\mathbf{n}}(\theta_1)\hat{\mathbf{c}} - \frac{\mathbf{y}_1}{\Gamma_2}, \\
\mathbf{x}_3 &= \frac{1}{\Gamma_3}(\Gamma_2\mathbf{x}_2-\mathbf{y}_2) = \frac{\Gamma_1}{\Gamma_3}R_{\mathbf{n}}(\theta_1)\hat{\mathbf{c}} - \frac{\mathbf{y}_1}{\Gamma_3} - \frac{\mathbf{y}_2}{\Gamma_3}, \\
&\dots \\
\mathbf{x}_{N-1} &= \frac{1}{\Gamma_{N-1}}(\Gamma_{N-2}\mathbf{x}_{N-2}-\mathbf{y}_{N-2}) = \frac{\Gamma_1}{\Gamma_{N-1}}R_{\mathbf{n}}(\theta_1)\hat{\mathbf{c}} - \sum_{i=1}^{N-2}\frac{\mathbf{y}_i}{\Gamma_{N-2}}, \\
\mathbf{x}_N &= \frac{1}{\Gamma_N}\left(\mathbf{c}-\sum_{i=1}^{N-1}\Gamma_i\mathbf{x}_i\right) = \frac{\mathbf{c}}{\Gamma_N} - (N-1)\frac{\Gamma_1}{\Gamma_N}R_{\mathbf{n}}(\theta_1)\hat{\mathbf{c}} + \sum_{i=1}^{N-2}(N-2-i)\frac{\mathbf{y}_i}{\Gamma_N},
\end{aligned}$$

where $\hat{\mathbf{c}} = \frac{\mathbf{c}}{|\mathbf{c}|}$ and $R_{\mathbf{n}}(\theta)$ is a rotation matrix on angle θ about axis with direction $\mathbf{n}$. The direction $\mathbf{n}$ should be orthogonal to $\mathbf{c}$. In the reconstruction formulas we have 3 degrees of freedom (the reduced degrees). These degrees of freedom characterize orientation of the vector $\mathbf{c}$ and orientation of vector $\mathbf{n}$. These parameters can be taken from the unreduced configuration (if we start with it) or chosen arbitrarily.

From (1.3.1) the reduced Hamiltonian can be written as

$$h(\theta_1, \mathbf{y}_1, \ldots, \mathbf{y}_{N-2}) = H(\mathbf{x}_1(\theta_1, \mathbf{y}_1, \ldots, \mathbf{y}_{N-2}), \ldots, \mathbf{x}_N(\theta_1, \mathbf{y}_1, \ldots, \mathbf{y}_{N-2})) =$$
$$= \sum_{i,j=1, j\neq i}^{N} \frac{\Gamma_i \Gamma_j}{4\pi} \ln |\mathbf{x}_i - \mathbf{x}_j| =$$
$$= \sum_{i,j=1, j\neq i}^{N-1} \frac{\Gamma_i \Gamma_j}{4\pi} \ln \left| \left(\frac{\Gamma_1}{\Gamma_i} - \frac{\Gamma_1}{\Gamma_j} \right) R_{\mathbf{n}}(\theta_1) \hat{\mathbf{c}} - \sum_{k=1}^{i-1} \frac{\mathbf{y}_k}{\Gamma_i} + \sum_{k=1}^{j-1} \frac{\mathbf{y}_k}{\Gamma_j} \right| +$$
$$+ \sum_{i=1}^{N-1} \frac{\Gamma_i \Gamma_N}{4\pi} \ln \left| \frac{\mathbf{c}}{\Gamma_N} - (N-1) \frac{\Gamma_1}{\Gamma_N} R_{\mathbf{n}}(\theta_1) \hat{\mathbf{c}} + \sum_{k=1}^{N-2} (N-2-k) \frac{\mathbf{y}_k}{\Gamma_N} - \frac{\Gamma_1}{\Gamma_i} R_{\mathbf{n}}(\theta_1) \hat{\mathbf{c}} + \sum_{k=1}^{i-1} \frac{\mathbf{y}_k}{\Gamma_i} \right|,$$

where the projection (1.5.11) is used for $\mathbf{x}_i,\ i = 1, \ldots, N$.

In order to obtain the reduced Poisson bracket, we use the same projection and relations

$$\nabla_1 = \frac{\partial}{\partial \mathbf{x}_1} = \frac{\partial \theta_1}{\partial \mathbf{x}_1} \frac{\partial}{\partial \theta_1} + \sum_{j=1}^{N-2} \frac{\partial \mathbf{y}_j}{\partial \mathbf{x}_1} \frac{\partial}{\partial \mathbf{y}_j} = \frac{\partial (\hat{\mathbf{c}} \cdot \mathbf{x}_1)}{\partial \mathbf{x}_1} \frac{\partial}{\sin\theta_1 \partial \theta_1} + \Gamma_1 \frac{\partial}{\partial \mathbf{y}_1} =$$
$$= \left(\hat{\mathbf{c}} \cdot (1,1,1) + \frac{1}{|\mathbf{c}|} (\Gamma_1, \Gamma_1, \Gamma_1) \cdot \mathbf{x}_1 + \frac{(\Gamma_1, \Gamma_1, \Gamma_1) \cdot \mathbf{c}}{|\mathbf{c}|^2} \hat{\mathbf{c}} \cdot \mathbf{x}_1 \right) \frac{\partial}{\sin\theta_1 \partial \theta_1} + \Gamma_1 \frac{\partial}{\partial \mathbf{y}_1} =$$
$$= \left(\hat{\mathbf{c}} \cdot (1,1,1) + \frac{1}{|\mathbf{c}|} (\Gamma_1, \Gamma_1, \Gamma_1) \cdot R_{\mathbf{n}}(\theta_1) \hat{\mathbf{c}} + \frac{(\Gamma_1, \Gamma_1, \Gamma_1) \cdot \mathbf{c}}{|\mathbf{c}|^2} \cos\theta_1 \right) \frac{\partial}{\sin\theta_1 \partial \theta_1} + \Gamma_1 \frac{\partial}{\partial \mathbf{y}_1},$$
$$\nabla_i = \left(\frac{1}{|\mathbf{c}|} (\Gamma_i, \Gamma_i, \Gamma_i) \cdot \mathbf{x}_i + \frac{(\Gamma_i, \Gamma_i, \Gamma_i) \cdot \mathbf{c}}{|\mathbf{c}|^2} \hat{\mathbf{c}} \cdot \mathbf{x}_i \right) \frac{\partial}{\sin\theta_1 \partial \theta_1} - \Gamma_i \frac{\partial}{\partial \mathbf{y}_{i-1}} + \Gamma_i \frac{\partial}{\partial \mathbf{y}_i} =$$
$$= \left(\frac{1}{|\mathbf{c}|} (\Gamma_i, \Gamma_i, \Gamma_i) + \frac{(\Gamma_i, \Gamma_i, \Gamma_i) \cdot \mathbf{c}}{|\mathbf{c}|^2} \hat{\mathbf{c}} \right) \cdot \left(\frac{\Gamma_1}{\Gamma_i} R_{\mathbf{n}}(\theta_1) \hat{\mathbf{c}} - \sum_{k=1}^{i-1} \frac{\mathbf{y}_k}{\Gamma_i} \right) \frac{\partial}{\sin\theta_1 \partial \theta_1} - \Gamma_i \frac{\partial}{\partial \mathbf{y}_{i-1}} + \Gamma_i \frac{\partial}{\partial \mathbf{y}_i},$$
$$i = 2, \ldots, N-2,$$
$$\nabla_{N-1} = \left(\frac{1}{|\mathbf{c}|} (\Gamma_{N-1}, \Gamma_{N-1}, \Gamma_{N-1}) \cdot \mathbf{x}_{N-1} + \frac{(\Gamma_{N-1}, \Gamma_{N-1}, \Gamma_{N-1}) \cdot \mathbf{c}}{|\mathbf{c}|^2} \hat{\mathbf{c}} \cdot \mathbf{x}_{N-1} \right) \frac{\partial}{\sin\theta_1 \partial \theta_1} -$$
$$-\Gamma_{N-1} \frac{\partial}{\partial \mathbf{y}_{N-2}} =$$
$$= \left(\frac{1}{|\mathbf{c}|} (\Gamma_{N-1}, \Gamma_{N-1}, \Gamma_{N-1}) + \frac{(\Gamma_{N-1}, \Gamma_{N-1}, \Gamma_{N-1}) \cdot \mathbf{c}}{|\mathbf{c}|^2} \hat{\mathbf{c}} \right) \cdot \left(\frac{\Gamma_1}{\Gamma_{N-1}} R_{\mathbf{n}}(\theta_1) \hat{\mathbf{c}} + \sum_{k=1}^{N-2} \frac{\mathbf{y}_k}{\Gamma_{N-1}} \right) \frac{\partial}{\sin\theta_1 \partial \theta_1} -$$
$$-\Gamma_{N-1} \frac{\partial}{\partial \mathbf{y}_{N-2}},$$
$$\nabla_N = \left(\frac{1}{|\mathbf{c}|} (\Gamma_N, \Gamma_N, \Gamma_N) \cdot \mathbf{x}_N + \frac{(\Gamma_N, \Gamma_N, \Gamma_N) \cdot \mathbf{c}}{|\mathbf{c}|^2} \hat{\mathbf{c}} \cdot \mathbf{x}_N \right) \frac{\partial}{\sin\theta_1 \partial \theta_1} =$$
$$= \left(\frac{1}{|\mathbf{c}|} (\Gamma_N, \Gamma_N, \Gamma_N) + \frac{(\Gamma_N, \Gamma_N, \Gamma_N) \cdot \mathbf{c}}{|\mathbf{c}|^2} \hat{\mathbf{c}} \right) \cdot \left(\frac{\mathbf{c}}{\Gamma_N} - (N-1) \frac{\Gamma_1}{\Gamma_N} R_{\mathbf{n}}(\theta_1) \hat{\mathbf{c}} + \sum_{k=1}^{N-2} (N-2-k) \frac{\mathbf{y}_k}{\Gamma_N} \right) \frac{\partial}{\sin\theta_1 \partial \theta_1}.$$

Then from (1.4.5) we have

$$
\begin{aligned}
&\{f,g\} = -R_{\mathbf{n}}(\theta_1)\hat{\mathbf{c}}\cdot \\
&\cdot\left[\left(\hat{\mathbf{c}}\cdot(1,1,1)+\frac{1}{|\mathbf{c}|}(\Gamma_1,\Gamma_1,\Gamma_1)\cdot R_{\mathbf{n}}(\theta_1)\hat{\mathbf{c}}+\frac{(\Gamma_1,\Gamma_1,\Gamma_1)\cdot\mathbf{c}}{|\mathbf{c}|^2}\cos\theta_1\right)\frac{\partial f}{\sin\theta_1\partial\theta_1}+\Gamma_1\frac{\partial f}{\partial\mathbf{y}_1}\right]\times \\
&\times\left[\left(\hat{\mathbf{c}}\cdot(1,1,1)+\frac{1}{|\mathbf{c}|}(\Gamma_1,\Gamma_1,\Gamma_1)\cdot R_{\mathbf{n}}(\theta_1)\hat{\mathbf{c}}+\frac{(\Gamma_1,\Gamma_1,\Gamma_1)\cdot\mathbf{c}}{|\mathbf{c}|^2}\cos\theta_1\right)\frac{\partial g}{\sin\theta_1\partial\theta_1}+\Gamma_1\frac{\partial g}{\partial\mathbf{y}_1}\right]- \\
&-\left(\sum_{i=2}^{N-1}\frac{\Gamma_1}{\Gamma_i}R_{\mathbf{n}}(\theta_1)\hat{\mathbf{c}}-\sum_{k=1}^{i-1}\frac{\mathbf{y}_k}{\Gamma_i}\right)\cdot \\
&\cdot\left[\left(\frac{1}{|\mathbf{c}|}(\Gamma_i,\Gamma_i,\Gamma_i)+\frac{(\Gamma_i,\Gamma_i,\Gamma_i)\cdot\mathbf{c}}{|\mathbf{c}|^2}\hat{\mathbf{c}}\right)\cdot\left(\frac{\Gamma_1}{\Gamma_i}R_{\mathbf{n}}(\theta_1)\hat{\mathbf{c}}-\sum_{k=1}^{i-1}\frac{\mathbf{y}_k}{\Gamma_i}\right)\frac{\partial f}{\sin\theta_1\partial\theta_1}-\Gamma_i\frac{\partial f}{\partial\mathbf{y}_{i-1}}+\Gamma_i\frac{\partial f}{\partial\mathbf{y}_i}\right]\times \\
&\left[\left(\frac{1}{|\mathbf{c}|}(\Gamma_i,\Gamma_i,\Gamma_i)+\frac{(\Gamma_i,\Gamma_i,\Gamma_i)\cdot\mathbf{c}}{|\mathbf{c}|^2}\hat{\mathbf{c}}\right)\cdot\left(\frac{\Gamma_1}{\Gamma_i}R_{\mathbf{n}}(\theta_1)\hat{\mathbf{c}}-\sum_{k=1}^{i-1}\frac{\mathbf{y}_k}{\Gamma_i}\right)\frac{\partial g}{\sin\theta_1\partial\theta_1}-\Gamma_i\frac{\partial g}{\partial\mathbf{y}_{i-1}}+\Gamma_i\frac{\partial g}{\partial\mathbf{y}_i}\right]- \\
&-\left(\frac{\mathbf{c}}{\Gamma_N}-(N-1)\frac{\Gamma_1}{\Gamma_N}R_{\mathbf{n}}(\theta_1)\hat{\mathbf{c}}+\sum_{i=1}^{N-2}(N-2-i)\frac{\mathbf{y}_i}{\Gamma_N}\right)\cdot \\
&\cdot\left[\left(\frac{1}{|\mathbf{c}|}(\Gamma_N,\Gamma_N,\Gamma_N)+\frac{(\Gamma_N,\Gamma_N,\Gamma_N)\cdot\mathbf{c}}{|\mathbf{c}|^2}\hat{\mathbf{c}}\right)\cdot\left(\frac{\mathbf{c}}{\Gamma_N}-(N-1)\frac{\Gamma_1}{\Gamma_N}R_{\mathbf{n}}(\theta_1)\hat{\mathbf{c}}+\sum_{k=1}^{N-2}(N-2-k)\frac{\mathbf{y}_k}{\Gamma_N}\right)\frac{\partial f}{\sin\theta_1\partial\theta_1}\right]\times \\
&\times\left[\left(\frac{1}{|\mathbf{c}|}(\Gamma_N,\Gamma_N,\Gamma_N)+\frac{(\Gamma_N,\Gamma_N,\Gamma_N)\cdot\mathbf{c}}{|\mathbf{c}|^2}\hat{\mathbf{c}}\right)\cdot\left(\frac{\mathbf{c}}{\Gamma_N}-(N-1)\frac{\Gamma_1}{\Gamma_N}R_{\mathbf{n}}(\theta_1)\hat{\mathbf{c}}+\sum_{k=1}^{N-2}(N-2-k)\frac{\mathbf{y}_k}{\Gamma_N}\right)\frac{\partial g}{\sin\theta_1\partial\theta_1}\right].
\end{aligned}
$$

Chapter 2

Relative equilibria

We start the chapter with the definition of the relative equilibria we study and the method of finding them. Then we give small primer on singular value decomposition and Shannon entropy, which we use to verify and classify the equilibria. In the last sections we find symmetric and asymmetric relative equilibrium configuration of point vortices.

2.1 Configuration matrix

Definition 2.1.1. We say that configuration of N point vortices on a sphere is a relative equilibrium if the evolution of the system can be represented as rotations of initial configuration.

In other words, system is not changing modulo natural action of group $SO(3)$, i.e. trajectories of the point vortices are orbits of the group. The equivalent condition is to require the distances between the vortices to be constant in time.

From the definition, we see that relative equilibria are fixed points of the equations of motion (1.2.3), i.e.

$$\sum_{k=1,k\neq i,k\neq j}^{N} \frac{\Gamma_k V_{ijk}}{\pi} \left(\frac{1}{l_{jk}^2} - \frac{1}{l_{ik}^2} \right) = 0, \tag{2.1.1}$$

This can be rewritten as

$$\begin{pmatrix} 0 & 0 & C_{123} & C_{124} & \dots & C_{12N} \\ 0 & C_{132} & 0 & C_{134} & \dots & C_{13N} \\ 0 & C_{142} & C_{143} & 0 & \dots & C_{14N} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{(N-1)N1} & C_{(N-1)N2} & C_{(N-1)N3} & C_{(N-1)N4} & \dots & 0 \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \dots \\ \Gamma_N \end{pmatrix} = \mathbf{0}. \tag{2.1.2}$$

or

$$\mathbf{A\Gamma} = \mathbf{0}, \tag{2.1.3}$$

where matrix $\mathbf{A}$ is $N \times \frac{N(N-1)}{2}$ matrix with components $A_{mk} = C_{ijk} = \frac{V_{ijk}}{\pi}\left(\frac{1}{l_{jk}^2} - \frac{1}{l_{ik}^2}\right)$ and m is an index number which corresponds to permutation (i, j). Vector $\mathbf{\Gamma} = (\Gamma_1, \dots, \Gamma_N)$ is a vector of intensities.

Definition 2.1.2. Matrix $\mathbf{A}$ is called a *configuration matrix* of the system of point vortices[1].

From (2.1.2) we see, that the question about existence of relative equilibrium for given geometric configuration can be reformulated as linear algebra question: *for a given matrix* $\mathbf{A}$ *find whether it has nontrivial null space*.

Since $\mathbf{A}$ has nontrivial null space if and only if $\mathbf{AA}^T$ has nontrivial null space, the necessary and sufficient condition for matrix $\mathbf{A}$ to have nontrivial equilibria is

$$\det(\mathbf{AA}^T) = 0. \tag{2.1.4}$$

[1] This definition and the approach we use to characterize relative equilibria of point vortices was introduced by P. Newton in [New01] and was used in [JN06, MON10] to classify symmetric relative equilibrium configurations.

As it is shown in section 1.5, the dimension of the reduced space is $(2N-3)$. Thus it is enough to consider only $2N-3$ independent variables. Since the distances l_{ij} are unsigned (they appear in squared form), it is easy to see that for the configurations where we have all of the point vortices in one hemisphere, we can not to distinguish this configuration from its mirror symmetry. This produces a singularity in any chart of the reduced space defined in terms of l_{ij}. Thus we can not use just l_{ij} as a coordinates on the reduced space. But we can use them as coordinates which describe relative equilibrium configuration, since we can use the same singularity to construct all of the relative equilibria described by distances.

In order to count independent distances needed to describe the relative equilibrium we use the following geometric observations: without loss of generality we can assume that one of the point vortices of relative equilibrium configuration is located at the north pole. Additionally, we can choose the next point vortex to be on 0-th longitude. Then in order to describe two point vortex relative equilibrium it is enough to provide one distance l_{12}. Then without loss of generality, we can assume the third votes will lie in the hemisphere from 0th to 180th longitude and we need exactly 2 distances l_{13} and l_{23} to describe the relative equilibrium configuration. For the fourth vortex we need all 3 distances l_{14}, l_{24} and l_{34} in order to describe the configuration. Every other point vortex need also at least 3 distances to describe its position. Thus, we will need $3N-3-2-1=3N-6=3(N-2)$ independent distances. We denote them $d_i,\ i=1,\dots,3N-6$.

As it is shown in Appendix A formula (A.0.6) gives relation between the distances. Since volumes are nonnegative, the expressions under the roots are nonnegative as well. Using simple algebraic technique we can get rid of the roots and at the end will obtain algebraic equations of order $2^{\text{number of roots}}=2^5=32$. These equations will represent algebraic restrictions for the distances. From the geometric observations, we know that these equations can be solved for d_i.

Appendix A also provides formula (A.0.2), which is a representation of V_{ijk} in terms of l_{ij}, l_{ik}, and l_{jk} and thus in terms of d_i. The condition (2.1.4) along with relations on the distances (A.0.6) represents an algebraic variety in the space of coordinates d_i.

Let

$$P(d_1, \ldots, d_{3N-6}) = \det(\mathbf{A}\mathbf{A}^T). \tag{2.1.5}$$

Then P is a continuous function, as a superposition of polynomial and a root function. Since $\mathbf{A}\mathbf{A}^T$ is a positive definite matrix, the only zeros of function P are those were it touches the abscise axis.

2.2 SVD and Shannon entropy

In order to find the dimension of the null space we will use method of singular value decomposition. Additionally, using notion of Shannon entropy we characterize found relative equilibria.

The most comprehensive decomposition of real or complex valued $N \times M$ matrix $\mathbf{A}$ is the singular value decomposition[2]. The N singular values, $\sigma^{(i)}$ ($i = 1, \ldots N$), of $\mathbf{A}$, are non-negative real numbers that satisfy

$$\mathbf{A}\mathbf{v}^{(i)} = \sigma^{(i)}\mathbf{u}^{(i)}; \quad \mathbf{A}^{\dagger}\mathbf{u}^{(i)} = \sigma^{(i)}\mathbf{v}^{(i)}, \tag{2.2.1}$$

The vector $\mathbf{u}^{(i)}$ is called the left-singular vector associated with $\sigma^{(i)}$, while $\mathbf{v}^{(i)}$ is the right-singular vector. In terms of these, the matrix $\mathbf{A}$ has the factorization

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\dagger} = \sum_{i=1}^{k} \sigma^{(i)}\mathbf{u}^{(i)}\mathbf{v}^{(i)T}, \quad (k \leq N) \tag{2.2.2}$$

[2]See [GVL96] for general theory on singular-value decomposition.

where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal (or unitary in the case if $\mathbf{A}$ is complex valued) and $\Sigma \in \mathbb{R}^{N\times N}$ is upper diagona. Here, the rank of $\mathbf{A}$ is k. The columns of U are the left-singular vectors $\mathbf{u}^{(i)}$, while the columns of V are the right-singular vectors $\mathbf{v}^{(i)}$. The matrix $\mathbf{\Sigma}$ is given by:

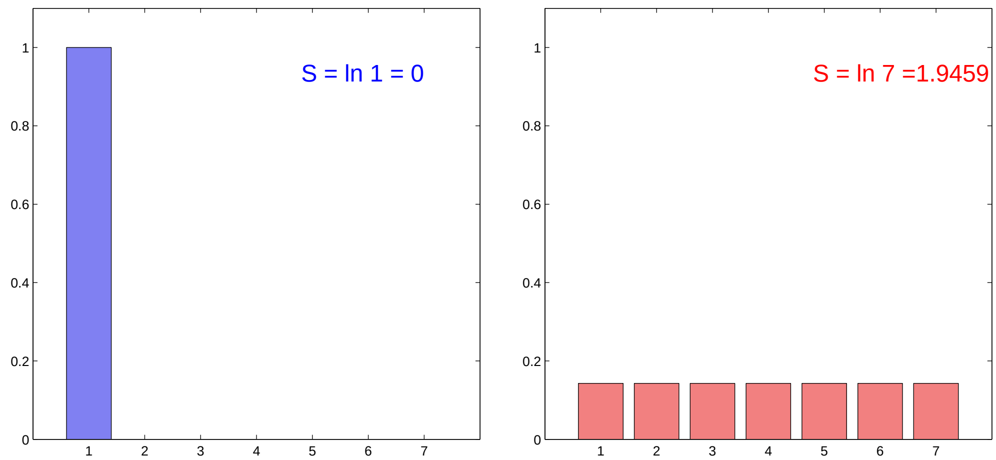
$$\mathbf{\Sigma} = \begin{pmatrix} \sigma^{(1)} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \sigma^{(N)} \\ \vdots & \dots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{N\times M}. \tag{2.2.3}$$

The singular values can be ordered so that $\sigma^{(1)} \geq \sigma^{(2)} \geq \ldots \geq \sigma^{(N)} \geq 0$ and one or more may be zero. As is evident from multiplying the first equation in (2.2.1) by $\mathbf{A}^\dagger$ and the second by $\mathbf{A}$,

$$(\mathbf{A}^\dagger\mathbf{A} - \sigma^{(i)2})\mathbf{v}^{(i)} = 0; \quad (\mathbf{A}\mathbf{A}^\dagger - \sigma^{(i)2})\mathbf{u}^{(i)} = 0, \tag{2.2.4}$$

the singular values squared are the eigenvalues of the *covariance* matrices $\mathbf{A}^\dagger\mathbf{A}$ or $\mathbf{A}\mathbf{A}^\dagger$, which have the same eigenvalue structure, while the left-singular vectors $\mathbf{u}^{(i)}$ are the eigenvectors of $\mathbf{A}\mathbf{A}^\dagger$, and the right-singular vectors $\mathbf{v}^{(i)}$ are the eigenvectors of $\mathbf{A}^\dagger\mathbf{A}$.

Since the set of singular values of matrix $\mathbf{A}$ characterizes matrix modulo orthogonal/unitary matrix multiplication it is customary to use the singular spectrum to characterize equilibrium configurations.



(a) (b)

Figure 2.1: Minimum (a) and maximum (b) entropy configuration for $N = 7$.

First we normalize each of the singular values so that they are positive and sum to one:

$$\hat{\sigma}^{(i)} \equiv \sigma^{(i)} / \sum_{j=1}^{k} \sigma^{(j)}. \tag{2.2.5}$$

Then

$$\sum_{i=1}^{k} \hat{\sigma}^{(i)} = 1, \tag{2.2.6}$$

and the string of k numbers arranged from largest to smallest: $(\hat{\sigma}^{(1)}, \hat{\sigma}^{(2)}, ..., \hat{\sigma}^{(k)})$ is the 'spectral representation' of the equilibrium. The rate at which they decay from largest to smallest is encoded in a scalar quantity called the Shannon entropy, S, of the matrix[3]:

$$S = -\sum_{i} \hat{\sigma}^{(i)} \ln \hat{\sigma}^{(i)}. \tag{2.2.7}$$

[3]See [SW48] and more recent discussions associated with vortex lattices in [CKN09].

With this representation, spectra that drop off rapidly from highest to lowest, are 'low-entropy equilibria', whereas those that drop off slowly (even distribution of normalized singular values) are 'high-entropy equilibria'. Note that from the representation (2.2.2), low-entropy equilibria have configuration matrix representations that are dominated in size by a small number of terms, whereas the configuration matrices of high-entropy equilibria equilibria have terms that are more equal in size. See [CKN09] for more detailed discussions in the context of relative equilibrium configurations, and the original report of [SW48] which has illuminating discussions of entropy, information content, and its interpretations with respect to randomness.

As an example of the normalized spectral distribution associated with the $N = 7$ singularities we show in Figure 2.1 the 7 singular values (including the zero one). The smallest value of the entropy is attained when all except one singular values are zeros. The non-zero singular value due to normalization should be 1. Then from (2.2.7) $S = 1 \ln 1 = 0$. The highest value of the entropy is attained when all of the singular values are non-zeros and equal, i.e. if the singular values are in the highest order. From (2.2.7) $S = -N \ln \frac{1}{N} = \ln N$.

Another useful application of SVD to point vortex problems is a method of Brownian ratchets. The method is used to find the asymmetric relative equilibrium configurations. The idea behind the method is minimization of the smallest singular value using random perturbation of positions of point vortices. The method was developed in [NC07] and later used in [NS09].

2.3 Symmetric relative equilibria

Symmetric configurations of point vortices on a sphere can easily be obtained from regular polyhedra. As it is shown in [Cox73], there are only five convex regular polyhedra in

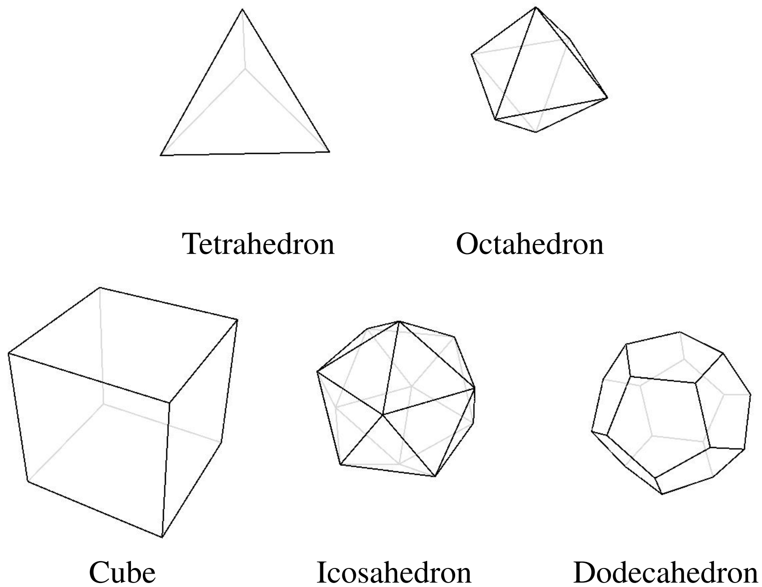
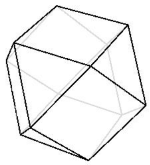


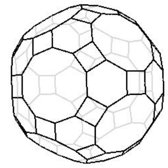
Figure 2.2: Platonic solids

3D: tetrahedron, octahedron, cube, icosahedron and dodecahedron. The family is called Platonic solids. One of the characteristic properties of the Platonic solids is transitivity of the faces, i.e. all of the faces of particular polyhedron can be described by a single regular polygon. Relaxation of this property, i.e. allowance for the faces to be different regular polygons, gives us Archimedean solids. Both, Platonic and Archimedean solids, are discrete subgroups of $\mathrm{SO}(3)$ and thus can be circumscribed into a sphere. By putting point vortices in the vertices of the polyhedra we can get the symmetric configurations of the point vortices on a sphere. Schematic wireframes of the Platonic and Archimedean solids are given on Figure 2.2 and Figure 2.3.

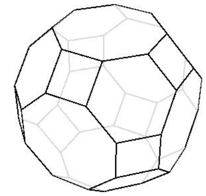
The general algebraic approach to the description of symmetric relative equilibria was first used in [LMR01]. There the authors considered discrete subgroups of $\mathrm{SO}(3)$ and proved existence of many symmetric relative equilibria of point vortices with equal strengths. Another approach was used by Jamaloodeen and Newton in [JN06]. They



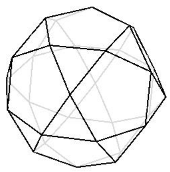
Cuboctahedron



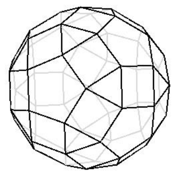
Great rhombicosidodecahedron



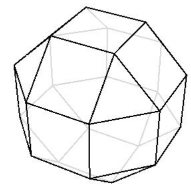
Great rhombicuboctahedron



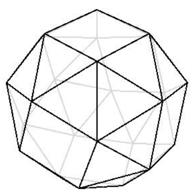
Icosidodecahedron



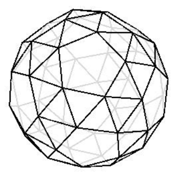
Small rhombicosidodecahedron



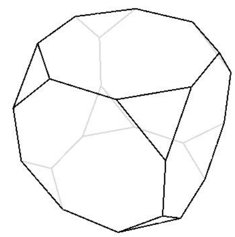
Small rhombicuboctahedron



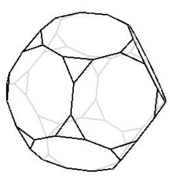
Snub cube



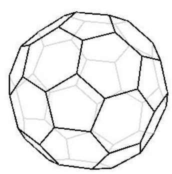
Snub dodecahedron



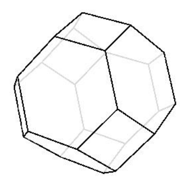
Truncated cube



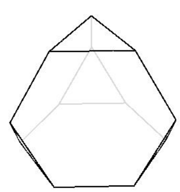
Truncated dodecahedron



Truncated icosahedron



Truncated octahedron



Truncated tetrahedron

Figure 2.3: Archimedean solids

used configuration matrix approach and proved that all of the Platonic solids are relative equilibria. Additionally they found the basis sets of the null spaces of the configuration matrices. Further extension of this work was done in [MON10], where the authors proved that among the Archimedean solids only cuboctahedron and icosidodecahedron are relative equilibrium configurations of point vortices on a sphere. In order to use in the stability study we will reproduce the results on Platonic and Archimedean solids with a small new result in terms of convenient and symmetric basis sets of the configuration matrices.

We use the following method to find the relative equilibrium configurations:

- For each of the symmetric configuration we compute the configuration matrix.
- Using SVD decomposition we find the singular values. If at least on of them is 0, then the configuration is a relative equilibrium.
- From SVD decomposition we find the dimension and the basis set of the null space of the configuration matrix. The basis vectors represent the intensities of point vortices which make the configuration a relative equilibrium.
- After normalization of singular values we compute the Shannon entropy, which is a useful characteristic of the configuration.

Tetrahedron.

Without loss of generality we can choose coordinates of point vortices in tetrahedral configuration to be

$$\mathbf{x}_1 = \frac{1}{\sqrt{3}}(1,1,1),\ \mathbf{x}_2 = \frac{1}{\sqrt{3}}(1,-1,-1), \tag{2.3.1}$$

$$\mathbf{x}_3 = \frac{1}{\sqrt{3}}(-1,1,-1),\ \mathbf{x}_4 = \frac{1}{\sqrt{3}}(-1,-1,1). \tag{2.3.2}$$

And since $l_{ij} = l_{lk}$ for any i, j, l, k, we have that $\mathbf{A} = \mathbf{0}$. Thus $\det(\mathbf{A}\mathbf{A}^T) = 0$ and tetrahedron is a relative equilibrium configuration for any choice of Γ_i.

Octahedron.

For octahedron we can choose coordinates of the vortices to be

$$\mathbf{x}_1 = (1, 0, 0),\ \mathbf{x}_2 = (-1, 0, 0),$$
$$\mathbf{x}_3 = (0, 1, 0),\ \mathbf{x}_4 = (0, -1, 0),$$
$$\mathbf{x}_5 = (0, 0, 1),\ \mathbf{x}_6 = (0, 0, -1).$$

Then $\mathbf{A} = \mathbf{0}$, as well, since either $l_{ij} = l_{lk}$ or $V_{ijk} = 0$. Thus, as above, for any Γ_i the configuration is a relative equilibrium.

Cube.

For a cubic configuration with vertices in

$$\begin{aligned} \mathbf{x}_1 = (0, 0, 1),\ \mathbf{x}_2 = (0, 0, -1), \\ \mathbf{x}_{i+3} = \left(\frac{2\sqrt{2}}{3}\cos\frac{2\pi i}{3}, \frac{2\sqrt{2}}{3}\sin\frac{2\pi}{3}, \frac{1}{3}\right),\ i = 0, .., 2, \\ \mathbf{x}_{i+6} = \left(\frac{2\sqrt{2}}{3}\cos\frac{\pi + 2\pi i}{3}, \frac{2\sqrt{2}}{3}\sin\frac{\pi + 2\pi}{3}, -\frac{1}{3}\right),\ i = 0, .., 2, \end{aligned} \tag{2.3.3}$$

Null-space of the configuration matrix is 5-dimensional. Basis for the null-space can be chosen as (see Figure 2.4)

$$\mathbf{b}_1 = (1, 1, 1, 1, 1, 1, 1, 1),$$
$$\mathbf{b}_2 = (1, -1, 0, 0, 0, 0, 0, 0),$$
$$\mathbf{b}_3 = (0, 0, 1, 0, 0, 0, -1, 0),$$
$$\mathbf{b}_4 = (0, 0, 0, 1, 0, 0, 0, -1),$$

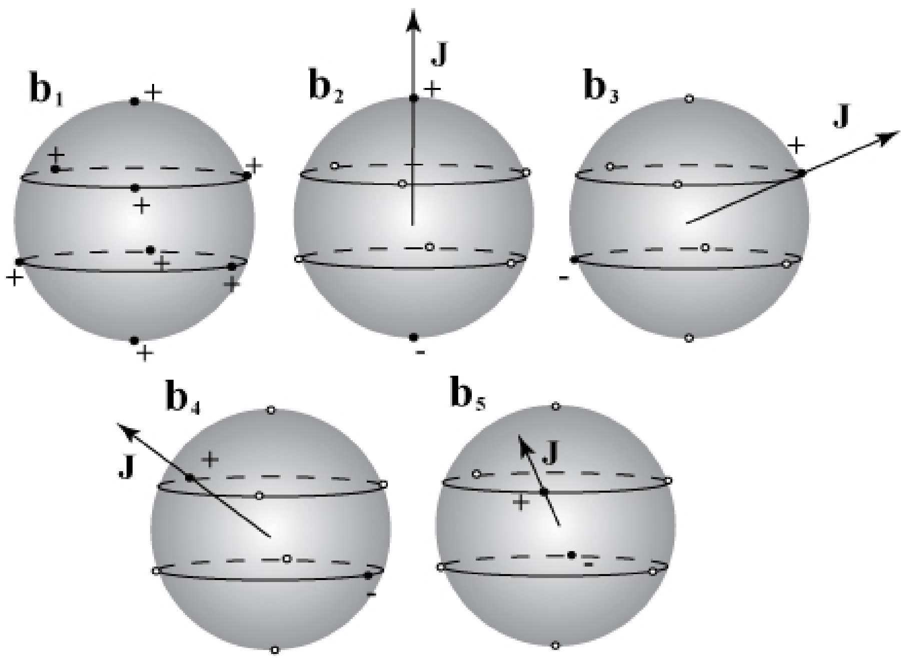


Figure 2.4: Null space of the configuration matrix for the cube

$$\mathbf{b}_5 = (0, 0, 0, 0, 1, -1, 0, 0).$$

Three non-zero singular values for the cube are equal to 4.

Icosahedron.

For the icosahedron with coordinates

$$\begin{aligned} \mathbf{x}_1 = (0,0,1),\ \mathbf{x}_2 = (0,0,-1), \\ \mathbf{x}_{i+3} = \left(\frac{2}{\sqrt{5}}\cos\frac{2\pi i}{5}, \frac{2}{\sqrt{5}}\sin\frac{2\pi}{5}, \frac{1}{\sqrt{5}}\right),\ i = 0,..,4, \\ \mathbf{x}_{i+8} = \left(\frac{2}{\sqrt{5}}\cos\frac{\pi + 2\pi i}{5}, \frac{2}{\sqrt{5}}\sin\frac{\pi + 2\pi}{5}, -\frac{1}{\sqrt{5}}\right),\ i = 0,..,4, \end{aligned} \tag{2.3.4}$$

configuration matrix $\mathbf{A}$ has seven dimensional null-space and basis for the null-space can be chosen as (see Figure 2.5)

$$\mathbf{b}_1 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$
$$\mathbf{b}_2 = (1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),$$
$$\mathbf{b}_3 = (0, 0, 1, 0, 0, 0, 0, 0, 0, -1, 0, 0),$$
$$\mathbf{b}_4 = (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, -1, 0),$$
$$\mathbf{b}_5 = (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, -1),$$
$$\mathbf{b}_6 = (0, 0, 0, 0, 0, 1, 0, -1, 0, 0, 0, 0),$$
$$\mathbf{b}_7 = (0, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0, 0).$$

Icosahedral configuration matrix has five singular values which are equal to

$$\sigma^{ico} = \sqrt{12(5 + \sqrt{5})}.$$

Dodecahedron.

For the dodecahedral configuration with point vortices at

$$\mathbf{x}_k = (\cos\frac{2k\pi}{5}s^u, \sin\frac{2k\pi}{5}s^u, z^u),\ k = 1, \ldots, 5, \tag{2.3.5}$$
$$\mathbf{x}_k = (\cos\frac{2k\pi}{5}s^l, \sin\frac{2k\pi}{5}s^l, z^l),\ k = 6, \ldots, 10, \tag{2.3.6}$$
$$\mathbf{x}_k = (\cos\frac{(2k+1)\pi}{5}s^l, \sin\frac{(2k+1)\pi}{5}s^l, -z^l),\ k = 11, \ldots, 15, \tag{2.3.7}$$
$$\mathbf{x}_k = (\cos\frac{(2k+1)\pi}{5}s^u, \sin\frac{(2k+1)\pi}{5}s^u, -z^u),\ k = 16, \ldots, 20, \tag{2.3.8}$$

(2.3.9)

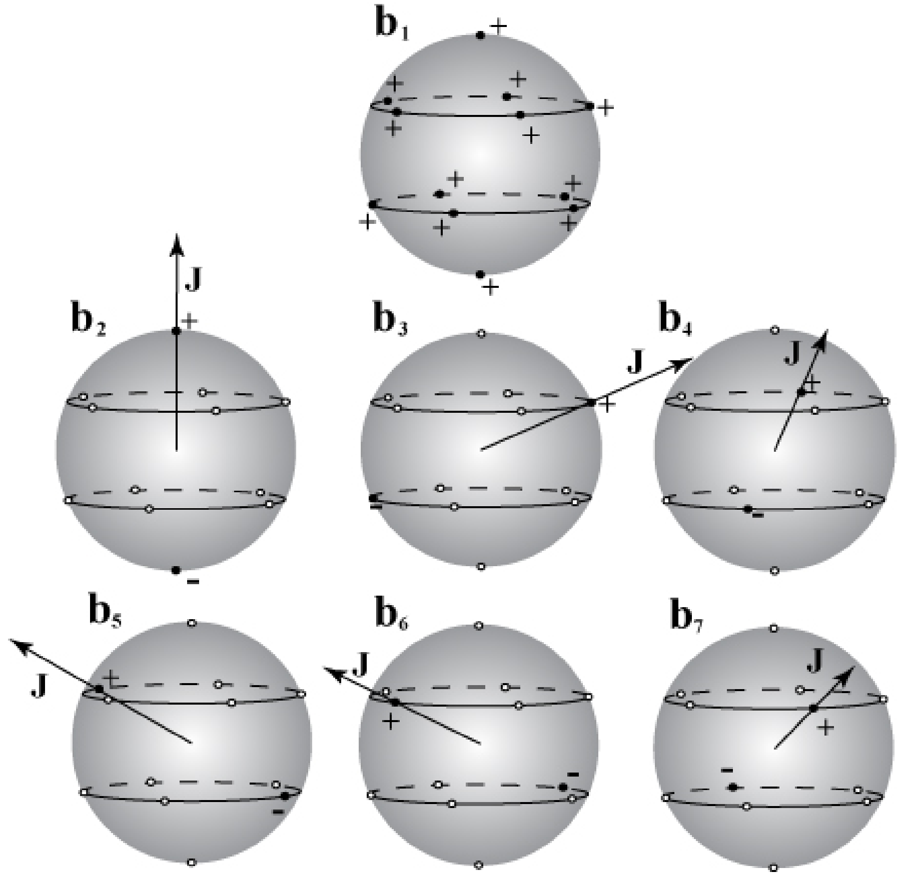


Figure 2.5: Null space of the configuration matrix for the icosahedron

where $z^u = \sqrt{\frac{1}{15}\left(5+2\sqrt{5}\right)}$, $z^l = \sqrt{\frac{1}{15}\left(5-2\sqrt{5}\right)}$, $s^u = \sqrt{\frac{2}{15}\left(5-\sqrt{5}\right)}$ and $s^l = \sqrt{\frac{2}{15}\left(5+\sqrt{5}\right)}$. The configuration matrix has 16 nonzero singular values, and 4 zeros (see Table 2.1). Thus the dimension of the null space is 4. The basis of the null space can be chosen to be symmetric with

$$\begin{aligned}
\mathbf{b}_1 &= (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1),\\
\mathbf{b}_2 &= (1+\phi,1+\phi,1+\phi,1+\phi,1+\phi,\phi,\phi,\phi,\phi,\phi,1,1,1,1,1,0,0,0,0,0),\\
\mathbf{b}_3 &= (1+\phi,\phi,1,\phi,1+\phi,1+\phi,1,0,1,1+\phi,\phi,0,0,\phi,1+\phi,1,0,0,1,\phi),\\
\mathbf{b}_4 &= (1+\phi,1+\phi,\phi,1,\phi,1+\phi,1+\phi,1,0,1,1+\phi,\phi,0,0,\phi,\phi,1,0,0,1),
\end{aligned}$$

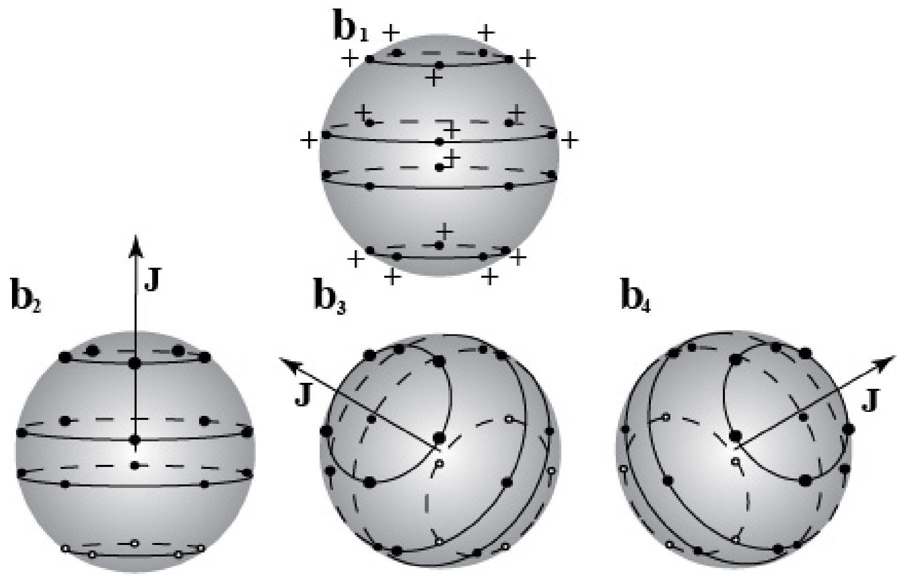


Figure 2.6: Null space of the configuration matrix for the dodecahedron

where $\phi = \frac{1+\sqrt{5}}{2}$ (golden ratio). Figure 2.6 gives a geometric description of the basis.

Cuboctahedron.

For cuboctahedron with point vortices at

$$\mathbf{x}_{i+j} = \frac{1}{\sqrt{2}}((-1)^i, (-1)^j, 0),\ i = 1, 3,\ j = 0, 1, \tag{2.3.10}$$

$$\mathbf{x}_{i+j} = \frac{1}{\sqrt{2}}((-1)^i, 0, (-1)^j),\ i = 5, 7,\ j = 0, 1, \tag{2.3.11}$$

$$\mathbf{x}_{i+j} = \frac{1}{\sqrt{2}}((-1)^i, (-1)^j, 0),\ i = 9, 11,\ j = 0, 1, \tag{2.3.12}$$

singular value decomposition of the configuration matrix has only one zero and 11 nonzero singular values. The null space contains only one vector

$$\mathbf{b}_1 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1).$$

Configuration	σ (unormalized)	σ (normalized)	Shannon entropy
Tetrahedron	0.0000, 0.0000, 0.0000, 0.0000	undefined undefined	undefined
Octahedron	0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000	undefined undefined undefined	undefined
Cube	4.0000, 4.0000, 4.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000	0.3333, 0.3333, 0.3333, 0.0000, 0.0000, 0.0000 0.0000, 0.0000	$\ln 3$=1.0986
Icosahedron	9.3184, 9.3184, 9.3184, 9.3184, 9.3184, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000	0.2000, 0.2000, 0.2000, 0.2000, 0.2000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000	$\ln 5 = 1.6094$
Dodecahedron	1.3270, 1.3270, 1.3270, 1.3270, 1.3270, 0.5324, 0.5324, 0.5324, 0.5324, 0.3550, 0.3550, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000	0.1728, 0.1728, 0.1728, 0.1728, 0.1728, 0.0278, 0.0278, 0.0278, 0.0278, 0.0124, 0.0124, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000	2.0241

Table 2.1: Singular spectra of Platonic solids relative equilibrium configurations

The distribution of singular values is given in Table 2.2.

Icosidodecahedron.

The icosidodecahedron has 30 vertices, which can be chosen to be

$$\mathbf{x}_i = (\pm 1, 0, 0),\ i = 1, 2,$$

$$\mathbf{x}_i = (0, \pm 1, 0),\ i = 3, 4,$$

$$\mathbf{x}_i = (0, 0, \pm 1),\ i = 5, 6,$$

$$\mathbf{x}_i = \frac{1}{\sqrt{2\phi^2+2\phi+2}}(\pm 1, \pm\phi, \pm(1+\phi)),\ i = 7, \ldots, 14,$$
$$\mathbf{x}_i = \frac{1}{\sqrt{2\phi^2+2\phi+2}}(\pm\phi, \pm(1+\phi), \pm 1),\ i = 15, \ldots, 22,$$
$$\mathbf{x}_i = \frac{1}{\sqrt{2\phi^2+2\phi+2}}(\pm(1+\phi), \pm 1, \pm\phi),\ i = 23, \ldots, 30.$$

The singular value decomposition of the configuration matrix has only one zero singular value. The only vector in the null space is a vector of equal intensities

$$\mathbf{b}_1 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1).$$

The distribution of singular values is given in Table 2.2.

Remaining Archimedean solids.

All of the other Archimedean solids configurations of point vortices have configuration matrices with a trivial null spaces, thus they do not represent relative equilibria.

Singular values and Shannon entropy.

Singular value distributions associated with each relative equilibrium configuration along with their Shannon entropy are given in Table 2.1 and Table 2.2.

Configuration	σ (unormalized)	σ (normalized)	Shannon entropy
Cuboctahedron	0.5381, 0.5381, 0.5381, 0.4660, 0.4660, 0.1272, 0.1272, 0.1272, 0.0734, 0.0734, 0.0734, 0.0000	0.2117, 0.2117, 0.2117, 0.1588, 0.1588, 0.0118, 0.0118, 0.0118, 0.0039, 0.0039, 0.0039, 0.0000	1.7934
Icosidodecahedron	2.6204, 2.6204, 2.6204, 2.6204, 2.6204, 1.1219, 1.1219, 1.1219, 1.1219, 0.9887, 0.9887, 0.9887, 0.5704, 0.5704, 0.5704, 0.5704, 0.3304, 0.3304, 0.3304, 0.3304, 0.3304, 0.2116, 0.2116, 0.2116, 0.2116, 0.2116, 0.1240, 0.1240, 0.1240, 0.0000	0.1546, 0.1546, 0.1546, 0.1546, 0.1546, 0.0283, 0.0283, 0.0283, 0.0283, 0.0220, 0.0220, 0.0220, 0.0073, 0.0073, 0.0073, 0.0073, 0.0025, 0.0025, 0.0025, 0.0025, 0.0025, 0.0010, 0.0010, 0.0010, 0.0010, 0.0010, 0.0003, 0.0003, 0.0003, 0.0000	2.3599

Table 2.2: Singular spectra of Archimedean solids relative equilibrium configurations

Chapter 3

Stability

We start this chapter with introduction to the energy-momentum method[1]. And then use the method to investigate stability of symmetric configurations, von Karman vortex streets and assymetric configurations.

3.1 Background

Let P be a phase space of Hamiltonian system with symplectic structure based on symplectic form Ω. Let $H : P \to \mathbb{R}$ be the Hamiltonian with vector field $X_H : P \to TP$, i.e.

$$DH(\mathbf{x}) \cdot \delta\mathbf{x} = \Omega(\mathbf{x})(X_H, \delta\mathbf{x}), \forall \mathbf{x} \in P, \delta\mathbf{x} \in T_{\mathbf{x}}P. \tag{3.1.1}$$

Let $F_t : [0, T] \times P \to P$ be the flow of vector field X_H. Then equations (3.1.1) can be rewritten as

$$\frac{d}{d\mathbf{x}} F_t(\mathbf{x}) = X(F(\mathbf{x})). \tag{3.1.2}$$

Let G be compact continuous symmetry group of the system, $\mathfrak{g}$ its Lie algebra, $\Psi_g : P \to P$ action of G on P for each $g \in G$. G is a symmetry group of the systems means that

$$H(\Psi_g(\mathbf{x})) = H(\mathbf{x}), \forall g \in G. \tag{3.1.3}$$

[1]More details on the energy-momentum method can be found in [MR99, SLM91].

For every action of G on P there is a corresponding vector field ξ_Q, which can be defined as

$$\xi_P(p) = \frac{d}{dt}(\exp[\epsilon\xi] \cdot p)_{\epsilon=0}, \quad (\xi, p) \in \mathfrak{g} \times P. \tag{3.1.4}$$

Let

$$\mathfrak{g} \cdot p = \{\xi_Q(q) | \xi \in \mathfrak{g}\} \subset T_p P, \tag{3.1.5}$$

be a tangent space to the orbit of the group $G \cdot p$. Assume that G acts freely on P (thus $G \cdot p \cong G$). This also means that $\xi_Q(q) = 0$ if and only if $\xi = 0$.

Let $\mathcal{F}(P) = \{f | f : P \to \mathbb{R}\}$. According to the Noether's theorem there is linear on symplectic leaves mapping $J : \mathfrak{g} \to \mathcal{F}(P)$ such as

$$X_{J(\xi)} = \xi_P, \forall \xi \in \mathfrak{g}, \tag{3.1.6}$$

Mapping $\mathbf{J} : P \to \mathfrak{g}^*$

$$J(\xi)(\mathbf{x}) =< \mathbf{J}(\mathbf{x}), \xi > \tag{3.1.7}$$

is called momentum map.

Now, we can define relative equilibrium using geometric approach. Point $\mathbf{x}_e \in P$ is called relative equilibrium of Hamiltonian system with symmetry G, if every trajectory which passes through $\mathbf{x}_e$ can be represented as

$$F_t(\mathbf{x}) = \Psi_{\exp[t\xi_e]}(\mathbf{x}_e), \tag{3.1.8}$$

for some $\xi_e \in \mathfrak{g}$. In other words, dynamical orbit which contains $\mathbf{x}_e$ coinsides with one-parametric orbit $\exp[t\xi_r]$.

If we differentiate (3.1.8) with respect to time, from equations of motion (3.1.2) and from (3.1.4) we will get

$$X_H(\mathbf{x}_e) = (\xi_e)_P(\mathbf{x}_e). \tag{3.1.9}$$

In order to find all of the relative equilibrium configurations of the Hamiltonian system with symmetry, we can use following theorem[2]

Theorem 3.1.1. $\mathbf{x}_e \in P$ *is a relative equilibrium of the dynamical system with Hamiltonian* H *with symmetry group* G *and momentum map* $\mathbf{J}$ *if and only if there is an* $\xi_e \in \mathfrak{g}$ *such that* $(\mathbf{x}_e, \xi_e)$ *is a critical point of energy-momentum functional* $H_{\mu_e} : P \times \mathfrak{g} \to \mathbb{R}$*:*

$$H_{\mu_e}(\mathbf{x}, \xi) = H(\mathbf{x}) - (\mathbf{J}(\mathbf{x}) - \mu_e) \cdot \xi, \tag{3.1.10}$$

where $\mu_e = \mathbf{J}(\mathbf{x}_e)$.

In other words, relative equilibria are critical points of Hamiltonian H restricted to the level set $\mathbf{J}^{-1}(\mu_e) \subset P$. Energy-momentum functional H_{μ_e} from Theorem 3.1.1 can be treated as Lagrange function in terms of optimization theory with the restrictions $\mathbf{J}(\mathbf{x}) - \mu_e = 0$ and ξ as Lagrange multipliers.

To study stability of the system we will look at the definiteness of the second variation of the energy-momentum functional. According to the method of Lagrange multipliers second variation in the restricted variational problem is strictly definite if it is strictly definite on the variations taken from the space of linearized restrictions. But for the Hamiltonian system with symmetries second variation is not definite even on that subspace, since Hamiltonian is invariant under the action of G. It will have neutral directions where the second variation is equal to zero. These directions will lie in the

[2] For details on the theorem and its proof see [AM78, Arn89].

intersection of $\mathfrak{g}\cdot\mathbf{x}_e$ (space tangent to $G\cdot\mathbf{x}_e$) and kernel of operator $T_{\mathbf{x}_e}\mathbf{J}$. A result from [MW74] shows that for equivariant momentum map

$$\mathfrak{g}\cdot\mathbf{x}_e\cap\ker[T_{\mathbf{x}_e}\mathbf{J}]=\mathfrak{g}_{\mu_e}, \tag{3.1.11}$$

where $\mathfrak{g}_{\mu_e}$ is a tangent space to $G_{\mu_e}\cdot\mathbf{x}_e$ and G_{μ_e} is an isotropy subgroup of μ_e. Lie algebra $\mathfrak{g}_{\mu_e}$ can be described as

$$\mathfrak{g}_{\mu_e}=\{\xi\in\mathfrak{g}|\mathrm{ad}^*_\xi\mu_e=0\}. \tag{3.1.12}$$

Notice, for any $\nu\in\mathfrak{g}$ and $\beta\in\mathfrak{g}_{\mu_e}$

$$\mathrm{ad}^*_\nu\mu_e\cdot\beta=\mu_e\cdot[\nu,\beta]=-\mu_e\cdot[\beta,\nu]=-\mathrm{ad}^*_\beta\mu_e\cdot\nu=0. \tag{3.1.13}$$

Equality (3.1.11) can be derived from the equivariance condition if we chose $g=\exp[\epsilon\xi]$ for arbitrary $\xi\in\mathfrak{g}$, then

$$\begin{aligned} T_{\mathbf{x}_e}\mathbf{J}\cdot\xi(\mathbf{x}_e) &\equiv \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\mathbf{J}(\Psi_{\exp[\epsilon\xi]}(\mathbf{x}_e))= \\ &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\mathrm{Ad}^*_{\exp[-\epsilon\xi]}(\mathbf{J}(\mathbf{x}_e))=\mathrm{ad}^*_\xi\mu_e. \end{aligned}$$

Thus $\xi_P(\mathbf{x}_e)\in\mathfrak{g}\cdot\mathbf{x}_e$ is a subspace of $\ker[T_{\mathbf{x}_e}\mathbf{J}]$ if and only if $\mathrm{ad}^*_\xi\mu_e=0$, or equivalently $\xi\in\mathfrak{g}_{\mu_e}$.

Since $H_{\mu_e}|_{\mathbf{J}^{-1}(\mu_e)\times\mathfrak{g}}$ is G_{μ_e} invariant

$$D^2H_{\mu_e}(\mathbf{x}_e,\xi_e)((\Delta\mathbf{x},0),(\delta\mathbf{x},0))=0$$

for any $\Delta\mathbf{x} \in \mathfrak{g}_{\mu_e} \cdot \mathbf{x}_e$ and $\delta\mathbf{x} \in T_{\mathbf{x}_e}\mathbf{J}^{-1}(\mu_e)$. Thus, relative equilibrium configuration can not be strict extremum of energy-momentum functional. But it can be strict extremum on the reduced space.

Theorem 3.1.2. [3] *Let $\mathbf{x}_e$ be a relative equilibrium with orbit $E = \{\exp[\xi_e t] \cdot \mathbf{x}_e, \forall t > 0\}$. E is a compact set in P. If energy-momentum functional H_{μ_e} attains its strict transversal to E extremum at the relative equilibrium $\mathbf{x}_e$, then orbit E is a stable relative equilibrium.*

This theorem allows one simplification. The simplification is based on introduction of notion of formal stability[4]

Definition 3.1.3. Relative equilibrium is formally stable if

$$D^2 H_{\mu_e}(\mathbf{x}_e, \xi_e)|_{\mathfrak{J}} \tag{3.1.14}$$

is definite. Space $\mathfrak{J}$ is a subspace of $\mathbf{J}^{-1}(\mu_e) \subset P$ (or equivalently $\mathfrak{J} \subset \ker[T_{\mathbf{x}_e}\mathbf{J}]$) which does not include neutrally stable directions.

From the definition of $\mathfrak{J}$ and representation (3.1.11) we have

$$\mathfrak{J} \cong \ker[T_{z_e}\mathbf{J}]/(\mathfrak{g}_{\mu_e} \cdot z_e). \tag{3.1.15}$$

Thus

$$\mathrm{codim}(\mathfrak{J}) = \mathrm{codim}(\ker[T_e\mathbf{J}]) + \dim(\mathfrak{g}_{\mu_e}). \tag{3.1.16}$$

The next theorem is a main theorem for the energy-momentum method and it shows that orbital stability (stability of relative equlibirum) follows from formal stability[5].

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[3] Proof of the theorem can be found in [Kur04, Kur05, KY02].

[4] In order to get more details see [AM78].

[5] The proof of the theorem can be found in [Pat92].

Theorem 3.1.4. *Assume $\mathbf{x}_e$ is a generic relative equilibrium with orbit*

$$E = \{\exp[\xi_e t] \cdot \mathbf{x}_e, \forall t > 0\}.$$

Assume also that G_{μ_e} is a proper action and $\mathfrak{g}$ admits inner product which is invariant under the action of G_{μ_e}. Then E is orbitally stable if $\mathbf{x}_e$ is a formally stable relative equilibrium.

Notice, relative equilibrium is generic if

$$\xi_P(\mathbf{x}_e) = \xi \cdot \mathbf{x}_e \neq 0, \forall \xi \in \mathfrak{g}. \tag{3.1.17}$$

Let us also recall that action of a group is proper, if Ψ_g is a proper mapping, i.e. preimage of compact set is a compact set.

Now let us apply the theory from the above to the problem of N point vortices on a sphere. At first we derive vector form of the method and then coordinate form.

Energy-momentum method in vector form

Since the Hamiltonian of the system of N point vortices is invariant under the action of $\mathrm{SO}(3)$ then $G = \mathrm{SO}(3)$. Action of G on is P is a rotation on each copy of S^2 in P. This action is a free canonical action on P. Lie algebra of G can be identified with R^3 with bracket

$$[\xi, \eta] = \boldsymbol{\xi} \times \boldsymbol{\eta}, \forall \xi, \eta \in \mathrm{so}(3).$$

Corresponding to $\xi \in \mathrm{so}(3)$ vector field is

$$\xi_P(\mathbf{x}) = \frac{d}{dt} \exp(\xi t) \cdot \mathbf{x}\bigg|_{t=0} = (\boldsymbol{\xi} \times \mathbf{x}_1, .., \boldsymbol{\xi} \times \mathbf{x}_N). \tag{3.1.18}$$

where $\mathbf{x} \in \mathbb{R}^{3N} = P$. As it was shown in first section, momentum map is

$$\mathbf{J}(\mathbf{x}) = -\sum_{i=1}^{N} \Gamma_i \mathbf{x}_i, \tag{3.1.19}$$

Using method of Lagrange multipliers, energy-momentum functional can be written as

$$\begin{aligned} H_{\mu_e}(\mathbf{x}_1, .., \mathbf{x}_N, \boldsymbol{\xi}) = \tfrac{1}{4\pi} \sum_{i<j} \Gamma_i \Gamma_j \ln(2(1 - \mathbf{x}_i \mathbf{x}_j)) + \\ + \sum_{i=1}^{N} [\Gamma_i (\mathbf{x}_i - \boldsymbol{\mu}_{e,i}) \cdot \boldsymbol{\xi} + k_i(||\mathbf{x}_i||^2 - 1)], \end{aligned} \tag{3.1.20}$$

To find μ which correspond to a relative equilibrium we use condition

$$\delta H_{\mu_e}(\mathbf{x}, \xi) = 0, \tag{3.1.21}$$

Or equivalently

$$\begin{aligned} \Gamma_i \boldsymbol{\xi} - \frac{\Gamma_i}{2\pi} \sum_{j=1, j\neq i}^{N} \Gamma_j \frac{\mathbf{x}_j}{l_{ij}^2} + 2k_i \mathbf{x}_i &= 0, \\ ||\mathbf{x}_i||^2 &= 1, \\ \mathbf{x}_i &= \boldsymbol{\mu}_{e,i}, \quad i = 1, .., N. \end{aligned} \tag{3.1.22}$$

If $\mathbf{J} \neq 0$, since it is an invariant of the system, all the vortices will rotate about $\mathbf{J}$. The symmetry group is $G_{\mu_e} = \mathrm{SO}(2)$. Orbits will be circles in the planes perpendicular to $\mathbf{J}$. Lie algebra is one-dimensional ($\dim(\mathrm{so}(2)) = 1$).

Derivative of momentum map can be written as

$$D\mathbf{J}(\mathbf{x}) \cdot \mathbf{y} = -\sum_{i=1}^{N} \Gamma_i \mathbf{y}_i, \tag{3.1.23}$$

where $\mathbf{y} = (\mathbf{y}_1, .., \mathbf{y}_N) \in T_{\mathbf{x}}P$. Thus

$$\ker D\mathbf{J}(\mathbf{x}) = \{\mathbf{y} \in T_{\mathbf{x}}P | \sum_{i=1}^{N} \Gamma_i \mathbf{y}_i = 0\}. \tag{3.1.24}$$

Second variation of energy momentum-functional is

$$D^2 H_{\mu_e} = \frac{\partial^2 H_{\mu_e}}{\partial a_i \partial b_j} = \begin{cases} \frac{2}{r} k_i \delta^{ab} - \frac{\Gamma_i}{\pi} \sum\limits_{k=1, k\neq i}^{N} \Gamma_k \frac{a_k b_k}{l_{ki}^4}, & i = j, \\ -\frac{\Gamma_i \Gamma_j}{2\pi l_{ij}^2} \left(\delta^{ab} + 2\frac{a_j b_i}{l_{ij}^2} \right), & i \neq j. \end{cases} \tag{3.1.25}$$

where $i, j = 1, .., N, a, b \in \{x, y, z\}$, $\delta^{ab} = 1$, when $a = b$ and $\delta^{ab} = 0$, when $a \neq b$. Since both $\mathrm{SO}(3)$ and $\mathrm{SO}(2)$ are compact, thus from all the conditions of the theorems 3.1.4,3.1.2 are satisfied and we can use (3.1.25) to prove stability.

If $\mathbf{J} = 0$ the symmetry group is $G_{\mu_e} = \mathrm{SO}(3)$. Tangent space to the orbit is a 3-dimensional space (since $\dim \mathrm{so}(3) = 3$). If we identify $\mathrm{so}(3)$ with $\mathbb{R}^3$ with vector product bracket then the tangent space is

$$\mathfrak{g}_{\mu_e} = \{(\mathbf{x} \times \mathbf{x}_1^e, \mathbf{x} \times \mathbf{x}_2^e, .., \mathbf{x} \times \mathbf{x}_N^e) | \mathbf{x} \in \mathbb{R}^3\}, \tag{3.1.26}$$

where $\mathbf{x}_e \in \mathbb{R}^{3N}, \mathbf{x}_e = (\mathbf{x}_1^e, \mathbf{x}_2^e, .., \mathbf{x}_N^e)$ is a relative equilibrium. Since

$$\sum_{i=1}^{N} \Gamma_i \mathbf{x} \times \mathbf{x}_i^e = \mathbf{x} \times \sum_{i=1}^{N} \Gamma_i \mathbf{x}_i^e = \mathbf{x} \times \mathbf{0} = \mathbf{0}, \tag{3.1.27}$$

thus from (3.1.24) we have $\mathfrak{g}_{\mu_e} \subset \ker D\mathbf{J}(\mathbf{0})$.

Complements $\mathfrak{C}_1 = \ker D\mathbf{J}(\mathbf{x}_e) \ominus \mathfrak{g}_{\mu_e}$ and $\mathfrak{C}_2 = TP \ominus \mathfrak{g}_{\mu_e}$ will be the transversal spaces which we need for the theorems 3.1.4,3.1.2. Good choice of the basis in these spaces allows us to simplify matrix of the second variation. This ”good choice” is completely dependent on the configuration.

Let B_1, B_2 be the basis sets for the spaces $\mathfrak{C}_1$ and $\mathfrak{C}_2$ accordingly. Second variation of H_{μ_e} along B_1, B_2 is

$$(\delta^2 H_{\mu_e}|_{B_l})_{i,j} = \delta\mathbf{x}_i^T D^2 H_{\mu_e} \delta\mathbf{x}_j, \quad \delta\mathbf{x}_i, \delta\mathbf{x}_j \in B_l,\ i,j = 1,..,N,\ l = 1,2. \quad (3.1.28)$$

Energy-momentum method in cylindrical coordinates

For the axis-symmetric configurations it is convenient to use cylindrical coordinates

$$z_i = \sqrt{\Gamma_i}\cos\theta_i,$$
$$\phi_i = \operatorname{sign}(\Gamma_i)\sqrt{\Gamma_i}\phi_i,$$
$$i = 1,..,N.$$

If $\mathbf{J} \neq 0$ we choose Oz to be aligned with $\mathbf{J}$. Then energy-momentum functional will be

$$H_{\mu_e} = \frac{1}{4\pi}\sum_{i<j}\Gamma_i\Gamma_j \ln[2(1 - z_i z_j - \sqrt{1-z_i^2}\sqrt{1-z_j^2}\cos(\phi_i - \phi_j))] - \omega\sum_{i=1}^{N}\Gamma_i z_i, \quad (3.1.29)$$

where ω is angular velocity of rotation about $\mathbf{J}$. Since the last term of the functional depends linearly on coordinates, second variation of H_{μ_e} coincides with second variation of H. For simplicity of computations let us multiply H by 4π. Then the components of $\bar{H} = 4\pi H$ are

$$\frac{\partial^2\bar{H}}{\partial\phi_i^2} =$$
$$= \sum_{j=1, j\neq i}^{N}\frac{\Gamma_i\Gamma_j}{2}\frac{\cos(\phi_i - \phi_j)\sqrt{1-z_i^2}\sqrt{1-z_j^2}(1 - z_i z_j) - (1 - z_i^2)(1 - z_j^2)}{l_{ij}^2},$$
$$\frac{\partial^2\bar{H}}{\partial\phi_i\partial\phi_j} =$$

$$= \frac{\Gamma_i \Gamma_j}{2} \frac{\cos(\phi_i - \phi_j)\sqrt{1-z_i^2}\sqrt{1-z_j^2}(1 - z_i z_j) + (1-z_i^2)(1-z_j^2)}{l_{ij}^2},$$

$$\frac{\partial^2 \bar{H}}{\partial z_i^2} =$$
$$= \sum_{j=1, j\neq i}^{N} \frac{\Gamma_i \Gamma_j}{2} \left[\frac{\left(z_j(1-z_i^2) - \cos(\phi_i - \phi_j) z_i \sqrt{1-z_j^2} \right)^2}{(1-z_i^2)\sqrt{1-z_j^2}\, l_{ij}^2} - \frac{\cos(\phi_i - \phi_j)}{(1-z_i^2) l_{ij}} \right],$$
$$\frac{\partial^2 \bar{H}}{\partial z_i \partial z_j} =$$
$$= \frac{\Gamma_i \Gamma_j}{2} \frac{\cos(\phi_i - \phi_j)\sqrt{1-z_i^2}\sqrt{1-z_j^2}(z_i z_j - 1) + \sqrt{1-z_i^2}\sqrt{1-z_j^2}}{\sqrt{1-z_i^2}\sqrt{1-z_j^2}\, l_{ij}^2},$$
$$\frac{\partial^2 \bar{H}}{\partial \phi_i^2} = \sum_{j=1, j\neq i}^{N} \frac{\Gamma_i \Gamma_j}{2} \frac{\sin(\phi_i - \phi_j)\sqrt{1-z_i^2}(z_i - z_j)}{\sqrt{1-z_j^2}\, l_{ij}^2},$$
$$\frac{\partial^2 \bar{H}}{\partial \phi_i \partial \phi_j} = \frac{\sin(\phi_i - \phi_j)\sqrt{1-z_i^2}(z_i - z_j)}{\sqrt{1-z_j^2}\, l_{ij}^2}.$$

If we know second variation of H_{μ_e} then matrix of linearized system can be found from

$$\frac{d}{dt} \begin{pmatrix} \mathbf{z} \\ \boldsymbol{\phi} \end{pmatrix} = \mathbf{L}_{2N} \begin{pmatrix} \mathbf{z} \\ \boldsymbol{\phi} \end{pmatrix} = (\boldsymbol{\Omega}^{\flat})^{-1} \mathbf{H}_{2N} \begin{pmatrix} \mathbf{z} \\ \boldsymbol{\phi} \end{pmatrix}, \tag{3.1.30}$$

where $(\boldsymbol{\Omega}^{\flat})^{-1}$ inverse of the symplectic form evaluated at the equilibrium and $\mathbf{z}, \boldsymbol{\phi}$ are coordinates of $z_i, \phi_i, i = 1, .., N$ and $\mathbf{H}_{2N} = D^2(\mathbf{z}_e, \boldsymbol{\phi}_e)$ is matrix of the second variation evaluated at the relative equilibrium.

Cylindrical coordinates on a sphere have 2 singularities at the poles. To remove this undesired property we introduce mixed atlas on P. For all the point vortices which are far enough from the poles we use cylindrical coordinates. For pole vortices we use cartesian coordinates $(x_{m,n}, y_{m,n})$ and $(x_{l,s}, y_{l,s})$ for those close to the north and south poles respectively. Let

$$\begin{aligned} (x_{l,s}, y_{l,s}, z_{l,s}), \\ z_{l,s} = -\sqrt{1 - x_{l,s}^2 - y_{l,s}^2}, \\ l = N+1, .., N+N_s, \end{aligned} \tag{3.1.31}$$

$$\begin{aligned} (x_{m,n}, y_{m,n}, z_{m,n}), \\ z_{m,n} = \sqrt{1 - x_{m,n}^2 - y_{m,n}^2}, \\ m = N+N_s+1, .., N+N_s+N_n, \end{aligned} \tag{3.1.32}$$

From the equations of motion we have

$$\begin{aligned} \dot{x}_{l,s} = \sum_{i=1}^{N} \frac{\Gamma_i}{2\pi l_{il}^2}(z_{l,s} \sin\phi_i \sqrt{1 - z_i^2} - y_{l,s} z_i) + \\ + \sum_{j=N+1, j\neq l}^{N+N_s} \frac{\Gamma_j}{2\pi l_{jl}^2}(y_{j,s} z_{l,s} - y_{l,s} z_{j,s}) + \\ + \sum_{k=N+N_s+1}^{N+N_s+N_n} \frac{\Gamma_k}{2\pi l_{kl}^2}(y_{k,n} z_{l,s} - y_{l,s} z_{k,n}), \end{aligned}$$

$$\begin{aligned} \dot{y}_{l,s} = \sum_{i=1}^{N} \frac{\Gamma_i}{2\pi l_{il}^2}(-z_{l,s} \cos\phi_i \sqrt{1 - z_i^2} + x_{l,s} z_i) + \\ + \sum_{j=N+1, j\neq l}^{N+N_s} \frac{\Gamma_j}{2\pi l_{jl}^2}(-x_{j,s} z_{l,s} + x_{l,s} z_{j,s}) + \\ + \sum_{k=N+N_s+1}^{N+N_s+N_n} \frac{\Gamma_k}{2\pi l_{kl}^2}(-x_{k,n} z_{l,s} + x_{l,s} z_{k,n}), \end{aligned}$$

and

$$\dot{x}_{m,n} = \sum_{i=1}^{N} \frac{\Gamma_i}{2\pi l_{im}^2}(z_{m,n}\sin\phi_i\sqrt{1-z_i^2} - y_{m,n}z_i) +$$

$$+ \sum_{j=N+1}^{N+N_s} \frac{\Gamma_j}{2\pi l_{jm}^2}(y_{j,s}z_{m,n} - y_{m,n}z_{j,s}) +$$

$$+ \sum_{k=N+N_s+1, k\neq m}^{N+N_s+N_n} \frac{\Gamma_k}{2\pi l_{km}^2}(y_{k,n}z_{m,n} - y_{m,n}z_{k,n}),$$

$$\dot{y}_{m,n} = \sum_{i=1}^{N} \frac{\Gamma_i}{2\pi l_{im}^2}(-z_{m,n}\cos\phi_i\sqrt{1-z_i^2} + x_{m,n}z_i) +$$

$$+ \sum_{j=N+1}^{N+N_s} \frac{\Gamma_j}{2\pi l_{jm}^2}(-x_{j,s}z_{m,n} + x_{m,n}z_{j,s}) +$$

$$+ \sum_{k=N+N_s+1, k\neq m}^{N+N_s+N_n} \frac{\Gamma_k}{2\pi l_{km}^2}(-x_{k,n}z_{m,n} + x_{m,n}z_{k,n}).$$

These equations can be written as

$$\dot{x}_{l,s} = -\frac{z_{l,s}}{\Gamma_s}\frac{\partial H}{\partial y_{l,s}}, \quad \dot{y}_{l,s} = \frac{z_{l,s}}{\Gamma_s}\frac{\partial H}{\partial x_{l,s}},$$
$$\dot{x}_{m,n} = -\frac{z_{m,n}}{\Gamma_n}\frac{\partial H}{\partial y_{m,n}}, \quad \dot{y}_{m,n} = \frac{z_{m,n}}{\Gamma_n}\frac{\partial H}{\partial x_{m,n}},$$

where H is a Hamiltonian in this mixed atlas

$$H = \frac{1}{4\pi}\Bigg(\sum_{1\leq i<j\leq N} \Gamma_i\Gamma_j \ln[2(1 - z_iz_j - \sqrt{1-z_i^2}\sqrt{1-z_j^2}\cos(\phi_i-\phi_j))] +$$
$$+ \sum_{i=1, j=N+1}^{i=N, j=N+N_s} \Gamma_i\Gamma_j \ln[2(1 - \sqrt{1-z_i^2}(x_{j,s}\cos\phi_i + y_{j,s}\sin\phi_i) - z_{j,s}z_i)] +$$

$$
\begin{aligned}
&+ \sum_{i=1,j=N+N_s+1}^{i=N,j=N+N_s+N_n} \Gamma_i\Gamma_j \ln[2(1-\sqrt{1-z_i^2}(x_{j,n}\cos\phi_i + y_{j,n}\sin\phi_i) - z_{j,n}z_i)] - \\
&+ \sum_{N+1\le i<j\le N+N_s+1} \Gamma_i\Gamma_j \ln[2(1 - x_{i,s}x_{j,s} - y_{i,s}y_{j,s} - z_{i,s}z_{j,s})] - \\
&+ \sum_{N+N_s+1\le i<j\le N+N_s+N_n+1} \Gamma_i\Gamma_j \ln[2(1 - x_{i,n}x_{j,n} - y_{i,n}y_{j,n} - z_{i,n}z_{j,n})] - \\
&+ \left. \sum_{i=N+1,j=N+N_s+1}^{i=N+N_s,j=N+N_s+N_n} \Gamma_i\Gamma_j \ln[2(1 - x_{i,s}x_{j,n} - y_{i,s}y_{j,n} - z_{i,s}z_{j,n})] \right) . \qquad (3.1.33)
\end{aligned}
$$

Notice that $z_{i,n}, z_{j,s}$ are the functions of $x_{i,s}, y_{i,s}, x_{j,n}, y_{j,n}$

$$
\begin{aligned}
z_{i,s} &= -\sqrt{1 - x_{i,s}^2 - y_{i,s}^2}, \\
z_{j,n} &= \sqrt{1 - x_{j,n}^2 - y_{j,n}^2}.
\end{aligned}
$$

Energy-momentum functional will have following form

$$
\begin{aligned}
H_{\mu_e} = H - \omega \left(\sum_{i=1}^{N} \Gamma_i z_i - \sum_{i=N+1}^{N+N_s} \Gamma_i \sqrt{1 - x_{i,s}^2 - y_{i,s}^2} + \right. \\
\left. + \sum_{i=N+N_s+1}^{N+N_s+N_n} \Gamma_i \sqrt{1 - x_{i,n}^2 - y_{i,n}^2} \right) . \qquad (3.1.34)
\end{aligned}
$$

Notice, that in contrast to simple cylindrical coordinates, last term of energy momentum functional is no longer linearly dependent on coordinates. Thus second variation of H_{μ_e} is different from the second variation of H.

If we have at most one vortex near the north pole and at most one near the south, i.e. if

$$
N_s \le 1, \quad N_n \le 1.
$$

the Hamiltonian (3.1.33) and energy-momentum functional (3.1.34) can be simplified to

$$H = \frac{1}{4\pi}\left(\sum_{1\le i<j\le N}\Gamma_i\Gamma_j \ln[2(1 - z_i z_j - \sqrt{1-z_i^2}\sqrt{1-z_j^2}\cos(\phi_i-\phi_j))]+ \right.$$
$$+\sum_{i=1}^{N}\Gamma_i\Gamma_s \ln[2(1-\sqrt{1-z_i^2}(x_{1,s}\cos\phi_i + y_{1,s}\sin\phi_i) - z_{1,s}z_i)] +$$
$$+\sum_{i=1}^{N}\Gamma_i\Gamma_n \ln[2(1-\sqrt{1-z_i^2}(x_{1,n}\cos\phi_i + y_{1,n}\sin\phi_i) - z_{1,n}z_i)] +$$
$$\left. +\Gamma_s\Gamma_n \ln[2(1 - x_{1,s}x_{1,n} - y_{1,s}y_{1,n} - z_{1,s}z_{1,n})]\right), \tag{3.1.35}$$

$$H_{\mu_e} = H - \omega\left(\sum_{i=1}^{N}\Gamma_i z_i - \Gamma_s\sqrt{1-x_{1,s}^2-y_{1,s}^2} + \Gamma_n\sqrt{1-x_{1,n}^2-y_{1,n}^2}\right), \tag{3.1.36}$$

where Γ_n, Γ_s are intensities of north and south polar vortices and $z_{1,s} = \sqrt{1-x_{1,s}^2-y_{1,s}^2}, \quad z_{1,n} = \sqrt{1-x_{1,n}^2-y_{1,n}^2}$.

Symmetry adapted basis

In order to simplify the computation of eigenvalues of the second variation of the energy-momentum functional, we will be using symmetry adapted basis whenever it will be available. The basis represents the invariant subspaces of the system symmetries.

Consider a subgroup G_d of permutation group S_N. Let the action $g \in G_d$ of the group be

$$g\cdot(x_1,\dots,x_N)^T = (x_{g(1)},\dots,x_{g(N)})^T. \tag{3.1.37}$$

There are irreducible representations $V_i \subset \mathbb{R}^N$, $i = 1,\dots,M$, such that

$$g\cdot V_i = V_i,\ \forall g \in G_d, \tag{3.1.38}$$

and let $\mathbf{v}_{i,j} \in V_i$ be the basis of the space V_i. Then the change of variables

$$(x_1, \ldots, x_N)^T = y_1 \mathbf{v}_{1,1} + \ldots + y_N \mathbf{v}_{k_M,M}, \tag{3.1.39}$$

where k_M is a number of basis vectors in V_M. This will give us a transformation matrix $W = (\mathbf{v}_{i,j})$. Since the transformation of coordinates is linear, the derivatives of the coordinate variables will transform according to the Jacobian W. And the second derivatives of the H_{μ_e} are

$$\frac{d^2 H_{\mu_e}}{d\mathbf{y}} = W^T \frac{d^2 H_{\mu_e}}{d\mathbf{x}} W. \tag{3.1.40}$$

Depending on the symmetry, some of the invariant subspaces will give us blocks of separated variables, which will allow us to block diagonalize the Hessian matrix.

3.2 Stability of polar vortex pair

Since motion of 2 point vortices is an integrable problem, we can prove stability of vortex pair using exact solution of the problem. To find the solution, consider equations of the motion in vector form

$$\begin{aligned} &\dot{\mathbf{x}}_i = \frac{\Gamma_j}{2\pi} \frac{\mathbf{x}_j \times \mathbf{x}_i}{(\mathbf{x}_i - \mathbf{x}_j)^2} = \frac{1}{2\pi l_{12}^2}(\Gamma_1 \mathbf{x}_1 + \Gamma_2 \mathbf{x}_2) \times \mathbf{x}_i = \frac{1}{2\pi l_{12}^2} \mathbf{J} \times \mathbf{x}_i, \\ &i = 1, 2,\ j = 1, 2,\ j \neq i, \\ &||x_i||^2 = 1,\ i = 1, 2. \end{aligned} \tag{3.2.1}$$

From (1.2.3) we have

$$\frac{dl_{12}^2}{dt} = 0.$$

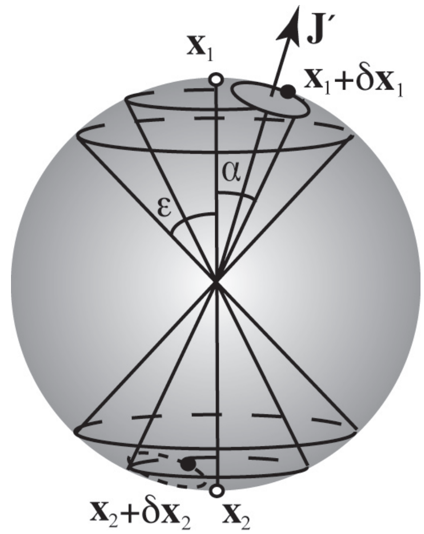


Figure 3.1: Perturbed degenerate vortex pair

Thus $l_{12}^2 = \mathrm{const}$ for any configuration of 2 point vortices and equations (3.2.1) describe rotations of $\mathbf{x}_i,\ i = 1, 2$ around constant vector $\frac{1}{2\pi l_{12}^2}\mathbf{J}$.

Theorem 3.2.1. *Degenerate vortex pair configuration (*$\mathbf{x}_1 = -\mathbf{x}_2$*,* $\Gamma_1 = -\Gamma_2$*) is stable equilibrium configuration.*

Proof. By adding small perturbations $\delta\mathbf{x}_1$ and $\delta\mathbf{x}_2$ to the given configuration we get nondegenerate vortex pair configuration. This configuration will rotate around fixed vector $\tilde{\mathbf{J}} = \Gamma_1(\mathbf{x}_1 + \delta\mathbf{x}_1) + \Gamma_2(\mathbf{x}_2 + \delta\mathbf{x}_2)$ and thus it will stay within spherical caps centered at $\mathbf{x}_1, \mathbf{x}_2$ with angular radius α equal to

$$\alpha = \cos^{-1}\left(\frac{\tilde{\mathbf{J}} \cdot \mathbf{J}}{||\tilde{\mathbf{J}}||||\mathbf{J}||}\right) + \max\{\cos^{-1}\left(\frac{\tilde{\mathbf{J}} \cdot (\mathbf{x}_i + \delta\mathbf{x}_i)}{||\tilde{\mathbf{J}}||||\mathbf{x}_i + \delta\mathbf{x}_i||}\right) \mid i = 1, 2\}.$$

Notice, that for any two perturbations $\delta\mathbf{x}_1, \delta\mathbf{x}_2$ from spherical caps with radius α vectors $\tilde{\mathbf{J}}, \mathbf{J}$ will stay inside one cap with radius α. Thus for any $\varepsilon > 0$ exists $\alpha = \varepsilon/3 > 0$, such that for any perturbations $\delta\mathbf{x}_1, \delta\mathbf{x}_2$ from spherical caps centered at $\mathbf{x}_1, \mathbf{x}_2$ with angular radius α perturbed system will stay within spherical caps centered at $\mathbf{x}_1, \mathbf{x}_2$ with radius ε. $\square$

Thus polar vortex pair with equal and opposite intensities is a stable configuration of point vortices.

3.3 Stability of tetrahedral configurations

As it was shown in previous section, tetrahedral configurations are relative equilibrium configuration for any choice of $\Gamma_i,\ i = 1, .., 4$. In this section we will start with stability of non-degenerate configurations ($\mathbf{J} \neq 0$).

Theorem 3.3.1. *Non-degenerate tetrahedral configurations are nonlinearly stable if*

$$\begin{aligned}
&\Gamma_3\Gamma_4(\Gamma_1^4\Gamma_2^2 + 2\Gamma_1^3\Gamma_2^3 + \Gamma_1^2\Gamma_2^4 + \Gamma_1^4\Gamma_2\Gamma_3 - 5\Gamma_1^3\Gamma_2^2\Gamma_3 - 5\Gamma_1^2\Gamma_2^3\Gamma_3 + \Gamma_1\Gamma_2^4\Gamma_3 + \\
&+\Gamma_1^3\Gamma_2\Gamma_3^2 + 2\Gamma_1^2\Gamma_2^2\Gamma_3^2 + \Gamma_1\Gamma_2^3\Gamma_3^2 + \Gamma_1^4\Gamma_2\Gamma_4 - 5\Gamma_1^3\Gamma_2^2\Gamma_4 - 5\Gamma_1^2\Gamma_2^3\Gamma_4 + \\
&+\Gamma_1\Gamma_2^4\Gamma_4 - 8\Gamma_1^3\Gamma_2\Gamma_3\Gamma_4 + 32\Gamma_1^2\Gamma_2^2\Gamma_3\Gamma_4 - 8\Gamma_1\Gamma_2^3\Gamma_3\Gamma_4 + \Gamma_1^3\Gamma_3^2\Gamma_4 - \\
&-5\Gamma_1^2\Gamma_2\Gamma_3^2\Gamma_4 - 5\Gamma_1\Gamma_2^2\Gamma_3^2\Gamma_4 + \Gamma_2^3\Gamma_3^2\Gamma_4 + \Gamma_1^3\Gamma_2\Gamma_4^2 + 2\Gamma_1^2\Gamma_2^2\Gamma_4^2 + \Gamma_1\Gamma_2^3\Gamma_4^2 + \\
&+\Gamma_1^3\Gamma_3\Gamma_4^2 - 5\Gamma_1^2\Gamma_2\Gamma_3\Gamma_4^2 - 5\Gamma_1\Gamma_2^2\Gamma_3\Gamma_4^2 + \Gamma_2^3\Gamma_3\Gamma_4^2 + \Gamma_1^2\Gamma_3^2\Gamma_4^2 + 2\Gamma_1\Gamma_2\Gamma_3^2\Gamma_4^2 + \\
&+\Gamma_2^2\Gamma_3^2\Gamma_4^2) > 0, \qquad (3.3.1)\\
&\sum_{i>j}\Gamma_i^2\Gamma_j^2 + \sum_{i\neq j\neq k}\Gamma_i^2\Gamma_j\Gamma_k - 30\Gamma_1\Gamma_2\Gamma_3\Gamma_4 > 0,\\
&\Gamma_1\Gamma_2(\Gamma_1^2(\Gamma_2+\Gamma_3) + \Gamma_2\Gamma_3(\Gamma_2+\Gamma_3) + \Gamma_1(\Gamma_2^2 - 6\Gamma_2\Gamma_3 + \Gamma_3^2)) \times\\
&\times(\Gamma_2\Gamma_3^2\Gamma_4(\Gamma_2\Gamma_3^2(2\Gamma_3 - \Gamma_4)\Gamma_4 + \Gamma_3^3\Gamma_4^2 + \Gamma_2^3(\Gamma_3+\Gamma_4)^2 + \\
&+\Gamma_2^2(\Gamma_3^3 - \Gamma_3\Gamma_4^2)) + \Gamma_1^4(\Gamma_3^2\Gamma_4(\Gamma_3+\Gamma_4)^2 + \Gamma_2^3(\Gamma_3^2 + 3\Gamma_3\Gamma_4 + \Gamma_4^2) + \\
&+\Gamma_2\Gamma_3(\Gamma_3^3 - 2\Gamma_3^2\Gamma_4 - 5\Gamma_3\Gamma_4^2 - 2\Gamma_4^3) + \Gamma_2^2(2\Gamma_3^3 + 2\Gamma_3\Gamma_4^2 + \Gamma_4^3)) + \\
&+\Gamma_1^2(\Gamma_3^4(2\Gamma_3 - \Gamma_4)\Gamma_4^2 - \Gamma_2\Gamma_3^3\Gamma_4(4\Gamma_3^2 - 8\Gamma_3\Gamma_4 + \Gamma_4^2) +
\end{aligned}$$

$$+\Gamma_2^3\Gamma_3(-4\Gamma_3^3+14\Gamma_3^2\Gamma_4+26\Gamma_3\Gamma_4^2-5\Gamma_4^3)+\Gamma_2^4(2\Gamma_3^3+2\Gamma_3\Gamma_4^2+\Gamma_4^3)+$$

$$+2\Gamma_2^2\Gamma_3^2(\Gamma_3^3+13\Gamma_3^2\Gamma_4-19\Gamma_3\Gamma_4^2+3\Gamma_4^3))+\Gamma_1\Gamma_3(\Gamma_3^4\Gamma_4^3-$$

$$-2\Gamma_2\Gamma_3^3\Gamma_4^2(2\Gamma_3+\Gamma_4)-\Gamma_2^2\Gamma_3^2\Gamma_4(4\Gamma_3^2-8\Gamma_3\Gamma_4+\Gamma_4^2)+$$

$$+\Gamma_2^4(\Gamma_3^3-2\Gamma_3^2\Gamma_4-5\Gamma_3\Gamma_4^2-2\Gamma_4^3)+\Gamma_2^3\Gamma_3(\Gamma_3^3-14\Gamma_3^2\Gamma_4-5\Gamma_3\Gamma_4^2+4\Gamma_4^3))+$$

$$+\Gamma_1^3(\Gamma_3^3\Gamma_4(\Gamma_3^2-\Gamma_4^2)+\Gamma_2^4(\Gamma_3^2+3\Gamma_3\Gamma_4+\Gamma_4^2)+\Gamma_2^2\Gamma_3(-4\Gamma_3^3+14\Gamma_3^2\Gamma_4+$$

$$+26\Gamma_3\Gamma_4^2-5\Gamma_4^3)-2\Gamma_2^3(2\Gamma_3^3+12\Gamma_3^2\Gamma_4+10\Gamma_3\Gamma_4^2-\Gamma_4^3)+$$

$$+\Gamma_2\Gamma_3^2(\Gamma_3^3-14\Gamma_3^2\Gamma_4-5\Gamma_3\Gamma_4^2+4\Gamma_4^3)))>0.$$

Proof. To prove this theorem we will use vector form of energy-momentum method. Since $\mathbf{J}\neq 0$, thus $G=\mathrm{SO}(2)$(i.e. symmetries are rotations about vector $\mathbf{J}$). Tangent to the orbit space is $\mathfrak{g}_{\mu_e}=\mathrm{span}\{\mathbf{y}_o=(\mathbf{J}\times\mathbf{x}_1,\mathbf{J}\times\mathbf{x}_2,\mathbf{J}\times\mathbf{x}_3,\mathbf{J}\times\mathbf{x}_4)\}$. To find $\mathfrak{C}_1$ we choose two linearly independent vectors $\mathbf{v}_1,\mathbf{v}_2\in\mathbb{R}^3$ in each copy of S^2. Lets also choose these vectors in such a way, that the vectors $\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4,\mathbf{J}$ and vectors $\mathbf{v}_1,\mathbf{v}_2$ are not co-planar. Consider following basis for the $T_{\mathbf{x}_e}P$:

$$\mathbf{e}^{(1)}=(\mathbf{x}_1\times\mathbf{v}_1,\mathbf{0},\mathbf{0},\mathbf{0}),$$
$$\mathbf{e}^{(2)}=(\mathbf{0},\mathbf{x}_2\times\mathbf{v}_1,\mathbf{0},\mathbf{0}),$$
$$\mathbf{e}^{(3)}=(\mathbf{0},\mathbf{0},\mathbf{x}_3\times\mathbf{v}_1,\mathbf{0}),$$
$$\mathbf{e}^{(4)}=(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{x}_4\times\mathbf{v}_1),$$
$$\mathbf{e}^{(5)}=(\mathbf{x}_1\times\mathbf{v}_2,\mathbf{0},\mathbf{0},\mathbf{0}),$$
$$\mathbf{e}^{(6)}=(\mathbf{0},\mathbf{x}_2\times\mathbf{v}_2,\mathbf{0},\mathbf{0}),$$
$$\mathbf{e}^{(7)}=(\mathbf{0},\mathbf{0},\mathbf{x}_3\times\mathbf{v}_2,\mathbf{0}),$$
$$\mathbf{e}^{(8)}=(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{x}_3\times\mathbf{v}_2),$$

Let $\alpha_i, i = 1, .., 4\ \ \beta_i, i = 1, .., 4$ be the solutions of

$$\begin{cases} \sum_{i=1}^{4} \Gamma_i \alpha_i \mathbf{x}_i = \mathbf{v}_1, \text{ or } \sum_{i=1}^{4} \Gamma_i \alpha_i \mathbf{x}_i = 0, \\ \sum_{i=1}^{4} \Gamma_i \beta_i \mathbf{x}_i = \mathbf{v}_2, \text{ or } \sum_{i=1}^{4} \Gamma_i \beta_i \mathbf{x}_i = 0. \end{cases} \tag{3.3.2}$$

Then $\mathbf{y} = \sum_{i=1}^{4} \alpha_i \mathbf{e}^{(i)} + \sum_{i=1}^{4} \beta_i \mathbf{e}^{(i+4)}$ will belong to $\ker D\mathbf{J}(\mathbf{x}_e)$, since

$$D\mathbf{J} \cdot \mathbf{y} = -\frac{1}{r} \left(\sum_{i=1}^{4} \Gamma_i \alpha_i \mathbf{x}_i \times \mathbf{v}_1 + \sum_{i=1}^{4} \Gamma_i \beta_i \mathbf{x}_i \times \mathbf{v}_2 \right) = 0,$$

From $\mathbf{J} \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ we have

$$\mathbf{J} \notin K = \operatorname{span}\{\sum_{i=1}^{4} \alpha_i \mathbf{e}^{(i)} + \sum_{i=1}^{4} \beta_i \mathbf{e}^{(i+4)} | \alpha_i, \beta_i \text{ - solutions of (3.3.2)}\}.$$

From (3.1.24) we have that dimension of $\ker D\mathbf{J}(\mathbf{x}_e)$ is $2N-3 = 8-3 = 5$. Dimension of $\mathfrak{g}_{\mu_e}$ is 1, thus $\dim(\mathfrak{C}_1) = 5 - 1 = 4$. Since every equation in (3.3.2) has $N - 2 = 2$ linearly independent solution and vectors $\mathbf{v}_1$, $\mathbf{v}_2$ are linearly independent we obtain $K = \mathfrak{C}_1$.

Let $\mathbf{v}_1 = \Gamma_1 \mathbf{x}_1 + \Gamma_2 \mathbf{x}_2 + \Gamma_3 \mathbf{x}_3$ and $\mathbf{v}_1 = \Gamma_1 \mathbf{x}_1 + \Gamma_2 \mathbf{x}_2 + \Gamma_4 \mathbf{x}_4$. Since $\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \neq 0$, thus $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{J} \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. One of the simplest solutions of the system (3.3.2) are

$$\alpha_1^{(1)} = 1,\ \alpha_2^{(1)} = 1,\ \alpha_3^{(1)} = 1, \alpha_4^{(1)} = 0,\ \beta_i^{(1)} = 0,\ i = 1, .., 4.$$
$$\beta_1^{(2)} = 1,\ \beta_2^{(2)} = 1,\ \beta_3^{(2)} = 0, \beta_4^{(2)} = 1,\ \alpha_i^{(2)} = 0,\ i = 1, .., 4.$$
$$\alpha_1^{(3)} = \frac{1}{\Gamma_1},\ \alpha_2^{(3)} = \frac{1}{\Gamma_2},\ \alpha_3^{(3)} = \frac{1}{\Gamma_3}, \alpha_4^{(3)} = \frac{1}{\Gamma_4},\ \beta_i^{(3)} = 0,\ i = 1, .., 4.$$

$$\beta_1^{(4)} = \frac{1}{\Gamma_1},\ \beta_2^{(4)} = \frac{1}{\Gamma_2},\ \beta_3^{(4)} = \frac{1}{\Gamma_3}, \beta_4^{(4)} = \frac{1}{\Gamma_4},\ \alpha_i^{(4)} = 0,\ i = 1, .., 4.$$

Thus basis of $\mathfrak{C}_1$ is

$$\begin{aligned}
&\boldsymbol{\epsilon}^{(1)} = \mathbf{e}^{(1)} + \mathbf{e}^{(2)} + \mathbf{e}^{(3)} = \\
&= (\Gamma_2 \mathbf{x}_1 \times \mathbf{x}_2 + \Gamma_3 \mathbf{x}_1 \times \mathbf{x}_3, -\Gamma_1 \mathbf{x}_1 \times \mathbf{x}_2 + \Gamma_3 \mathbf{x}_2 \times \mathbf{x}_3, \\
&-\Gamma_1 \mathbf{x}_1 \times \mathbf{x}_3 - \Gamma_2 \mathbf{x}_2 \times \mathbf{x}_3, \mathbf{0}), \\
&\boldsymbol{\epsilon}^{(2)} = \mathbf{e}^{(5)} + \mathbf{e}^{(6)} + \mathbf{e}^{(8)} = \\
&= (\Gamma_2 \mathbf{x}_1 \times \mathbf{x}_2 + \Gamma_4 \mathbf{x}_1 \times \mathbf{x}_4, -\Gamma_1 \mathbf{x}_1 \times \mathbf{x}_2 + \Gamma_4 \mathbf{x}_2 \times \mathbf{x}_4, \\
&\mathbf{0}, -\Gamma_1 \mathbf{x}_1 \times \mathbf{x}_4 - \Gamma_2 \mathbf{x}_2 \times \mathbf{x}_4), \\
&\boldsymbol{\epsilon}^{(3)} = \frac{1}{\Gamma_1}\mathbf{e}^{(1)} + \frac{1}{\Gamma_2}\mathbf{e}^{(2)} + \frac{1}{\Gamma_3}\mathbf{e}^{(3)} + \frac{1}{\Gamma_4}\mathbf{e}^{(4)} = \\
&= (\frac{\Gamma_2}{\Gamma_1}\mathbf{x}_1 \times \mathbf{x}_2 + \frac{\Gamma_3}{\Gamma_1}\mathbf{x}_1 \times \mathbf{x}_3, -\frac{\Gamma_1}{\Gamma_2}\mathbf{x}_1 \times \mathbf{x}_2 + \frac{\Gamma_3}{\Gamma_2}\mathbf{x}_2 \times \mathbf{x}_3, \\
&-\frac{\Gamma_1}{\Gamma_3}\mathbf{x}_1 \times \mathbf{x}_3 - \frac{\Gamma_2}{\Gamma_3}\mathbf{x}_2 \times \mathbf{x}_3, -\frac{\Gamma_1}{\Gamma_4}\mathbf{x}_1 \times \mathbf{x}_4 - \frac{\Gamma_2}{\Gamma_4}\mathbf{x}_2 \times \mathbf{x}_4 - \frac{\Gamma_3}{\Gamma_4}\mathbf{x}_3 \times \mathbf{x}_4), \\
&\boldsymbol{\epsilon}^{(4)} = \frac{1}{\Gamma_1}\mathbf{e}^{(5)} + \frac{1}{\Gamma_2}\mathbf{e}^{(6)} + \frac{1}{\Gamma_3}\mathbf{e}^{(7)} + \frac{1}{\Gamma_4}\mathbf{e}^{(8)} = \\
&= (\frac{\Gamma_2}{\Gamma_1}\mathbf{x}_1 \times \mathbf{x}_2 + \frac{\Gamma_4}{\Gamma_1}\mathbf{x}_1 \times \mathbf{x}_4, -\frac{\Gamma_1}{\Gamma_2}\mathbf{x}_1 \times \mathbf{x}_2 + \frac{\Gamma_4}{\Gamma_2}\mathbf{x}_2 \times \mathbf{x}_4, \\
&-\frac{\Gamma_1}{\Gamma_3}\mathbf{x}_1 \times \mathbf{x}_3 - \frac{\Gamma_2}{\Gamma_3}\mathbf{x}_2 \times \mathbf{x}_3 + \frac{\Gamma_4}{\Gamma_3}\mathbf{x}_3 \times \mathbf{x}_4, -\frac{\Gamma_1}{\Gamma_4}\mathbf{x}_1 \times \mathbf{x}_4 - \frac{\Gamma_2}{\Gamma_4}\mathbf{x}_2 \times \mathbf{x}_4).
\end{aligned}$$

If we choose the coordinates of the vertices

$$\begin{aligned}
\mathbf{x}_1 &= (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), \\
\mathbf{x}_2 &= (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), \\
\mathbf{x}_3 &= (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}),
\end{aligned}$$

$$\mathbf{x}_4 = (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}).$$

Then

$$\begin{aligned}
\mathbf{x}_1 \times \mathbf{x}_2 &= (0, \frac{4}{3}, -\frac{4}{3}), \\
\mathbf{x}_1 \times \mathbf{x}_3 &= (-\frac{4}{3}, 0, \frac{4}{3}), \\
\mathbf{x}_1 \times \mathbf{x}_4 &= (\frac{4}{3}, -\frac{4}{3}, 0), \\
\mathbf{x}_2 \times \mathbf{x}_3 &= (\frac{4}{3}, \frac{4}{3}, 0), \\
\mathbf{x}_2 \times \mathbf{x}_4 &= (-\frac{4}{3}, 0, -\frac{4}{3}), \\
\mathbf{x}_3 \times \mathbf{x}_4 &= (0, \frac{4}{3}, \frac{4}{3}).
\end{aligned}$$

From (3.1.22) we have

$$\begin{aligned}
k_i &= -\frac{3\Gamma_i^2}{32\pi}, \; i = 1, .., 4, \\
\boldsymbol{\xi} &= (\xi_1, \xi_2, \xi_3), \\
\xi_1 &= \frac{\sqrt{3}(\Gamma_1 + \Gamma_2 - \Gamma_3 - \Gamma_4)}{16\pi}, \\
\xi_2 &= \frac{\sqrt{3}(\Gamma_1 - \Gamma_2 + \Gamma_3 - \Gamma_4)}{16\pi}, \\
\xi_3 &= \frac{\sqrt{3}(\Gamma_1 - \Gamma_2 - \Gamma_3 + \Gamma_4)}{16\pi}.
\end{aligned}$$

Second variation of energy-momentum along $\boldsymbol{\epsilon}^{(j)}, \; j = 1, .., 4$ is

$$D^2 H_{\mu_e}|_{\boldsymbol{\epsilon}^{(j)}, j=1,..,4} = (d_{ij})_{i=1,..,4, j=1,..,4},$$

where

$$d_{11} = -\frac{1}{3}\Gamma_1^2\Gamma_2\Gamma_4 - \frac{1}{3}\Gamma_1\Gamma_2^2\Gamma_4 - \frac{1}{3}\Gamma_1^2\Gamma_3\Gamma_4 + 2\Gamma_1\Gamma_2\Gamma_3\Gamma_4 -$$
$$-\frac{1}{3}\Gamma_2^2\Gamma_3\Gamma_4 - \frac{1}{3}\Gamma_1\Gamma_3^2\Gamma_4 - \frac{1}{3}\Gamma_2\Gamma_3^2\Gamma_4,$$
$$d_{22} = -\frac{1}{3}\Gamma_1^2\Gamma_2\Gamma_3 - \frac{1}{3}\Gamma_1\Gamma_2^2\Gamma_3 - \frac{1}{3}\Gamma_1^2\Gamma_3\Gamma_4 + 2\Gamma_1\Gamma_2\Gamma_3\Gamma_4 -$$
$$-\frac{1}{3}\Gamma_2^2\Gamma_3\Gamma_4 - \frac{1}{3}\Gamma_1\Gamma_3\Gamma_4^2 - \frac{1}{3}\Gamma_2\Gamma_3\Gamma_4^2,$$
$$d_{33} = 2\Gamma_1^2 - \frac{4\Gamma_1\Gamma_2}{3} + 2\Gamma_2^2 - \frac{\Gamma_1^2\Gamma_2}{3\Gamma_3} - \frac{\Gamma_1\Gamma_2^2}{3\Gamma_3} - \frac{4\Gamma_1\Gamma_3}{3} - \frac{\Gamma_1^2\Gamma_3}{3\Gamma_2} - \frac{4\Gamma_2\Gamma_3}{3} -$$
$$-\frac{\Gamma_2^2\Gamma_3}{3\Gamma_1} + 2\Gamma_3^2 - \frac{\Gamma_1\Gamma_3^2}{3\Gamma_2} - \frac{\Gamma_2\Gamma_3^2}{3\Gamma_1} - \frac{\Gamma_1^2\Gamma_2}{3\Gamma_4} - \frac{\Gamma_1\Gamma_2^2}{3\Gamma_4} - \frac{\Gamma_1^2\Gamma_3}{3\Gamma_4} + \frac{2\Gamma_1\Gamma_2\Gamma_3}{\Gamma_4} -$$
$$-\frac{\Gamma_2^2\Gamma_3}{3\Gamma_4} - \frac{\Gamma_1\Gamma_3^2}{3\Gamma_4} - \frac{\Gamma_2\Gamma_3^2}{3\Gamma_4} - \frac{\Gamma_1^2\Gamma_4}{3\Gamma_2} - \frac{\Gamma_2^2\Gamma_4}{3\Gamma_1} - \frac{\Gamma_1^2\Gamma_4}{3\Gamma_3} + \frac{2\Gamma_1\Gamma_2\Gamma_4}{3\Gamma_3} - \frac{\Gamma_2^2\Gamma_4}{3\Gamma_3} +$$
$$+\frac{2\Gamma_1\Gamma_3\Gamma_4}{3\Gamma_2} + \frac{2\Gamma_2\Gamma_3\Gamma_4}{3\Gamma_1} - \frac{\Gamma_3^2\Gamma_4}{3\Gamma_1} - \frac{\Gamma_3^2\Gamma_4}{3\Gamma_2},$$

$$d_{44} = 2\Gamma_1^2 - \frac{4\Gamma_1\Gamma_2}{3} + 2\Gamma_2^2 - \frac{\Gamma_1^2\Gamma_2}{3\Gamma_3} - \frac{\Gamma_1\Gamma_2^2}{3\Gamma_3} - \frac{\Gamma_1^2\Gamma_3}{3\Gamma_2} - \frac{\Gamma_2^2\Gamma_3}{3\Gamma_1} - \frac{\Gamma_1^2\Gamma_2}{3\Gamma_4} -$$
$$-\frac{\Gamma_1\Gamma_2^2}{3\Gamma_4} - \frac{\Gamma_1^2\Gamma_3}{3\Gamma_4} + \frac{2\Gamma_1\Gamma_2\Gamma_3}{3\Gamma_4} - \frac{\Gamma_2^2\Gamma_3}{3\Gamma_4} - \frac{4\Gamma_1\Gamma_4}{3} - \frac{\Gamma_1^2\Gamma_4}{3\Gamma_2} - \frac{4\Gamma_2\Gamma_4}{3} - \frac{\Gamma_2^2\Gamma_4}{3\Gamma_1} -$$
$$-\frac{\Gamma_1^2\Gamma_4}{3\Gamma_3} + \frac{2\Gamma_1\Gamma_2\Gamma_4}{\Gamma_3} - \frac{\Gamma_2^2\Gamma_4}{3\Gamma_3} + \frac{2\Gamma_1\Gamma_3\Gamma_4}{3\Gamma_2} + \frac{2\Gamma_2\Gamma_3\Gamma_4}{3\Gamma_1} + 2\Gamma_4^2 - \frac{\Gamma_1\Gamma_4^2}{3\Gamma_2} -$$
$$-\frac{\Gamma_2\Gamma_4^2}{3\Gamma_1} - \frac{\Gamma_1\Gamma_4^2}{3\Gamma_3} - \frac{\Gamma_2\Gamma_4^2}{3\Gamma_3} - \frac{\Gamma_3\Gamma_4^2}{3\Gamma_1} - \frac{\Gamma_3\Gamma_4^2}{3\Gamma_2},$$

$$d_{12} = d_{21} = \frac{1}{3}\Gamma_1^2\Gamma_3\Gamma_4 - \frac{2}{3}\Gamma_1\Gamma_2\Gamma_3\Gamma_4 + \frac{1}{3}\Gamma_2^2\Gamma_3\Gamma_4,$$
$$d_{13} = d_{31} = \frac{1}{3}\Gamma_1^2\Gamma_2 + \frac{1}{3}\Gamma_1\Gamma_2^2 + \frac{1}{3}\Gamma_1^2\Gamma_3 - 2\Gamma_1\Gamma_2\Gamma_3 + \frac{1}{3}\Gamma_2^2\Gamma_3 + \frac{1}{3}\Gamma_1\Gamma_3^2 +$$
$$+\frac{1}{3}\Gamma_2\Gamma_3^2 - \frac{2}{3}\Gamma_1^2\Gamma_4 + \frac{2}{3}\Gamma_1\Gamma_2\Gamma_4 - \frac{2}{3}\Gamma_2^2\Gamma_4 + \frac{2}{3}\Gamma_1\Gamma_3\Gamma_4 + \frac{2}{3}\Gamma_2\Gamma_3\Gamma_4 - \frac{2}{3}\Gamma_3^2\Gamma_4,$$

$$d_{14}=d_{41}=\frac{1}{3}\Gamma_1^2\Gamma_2+\frac{1}{3}\Gamma_1\Gamma_2^2+\frac{1}{3}\Gamma_1^2\Gamma_3-\frac{4}{3}\Gamma_1\Gamma_2\Gamma_3+\frac{1}{3}\Gamma_2^2\Gamma_3-\frac{2}{3}\Gamma_1^2\Gamma_4+$$
$$+\frac{2}{3}\Gamma_1\Gamma_2\Gamma_4-\frac{2}{3}\Gamma_2^2\Gamma_4+\frac{1}{3}\Gamma_1\Gamma_3\Gamma_4+\frac{1}{3}\Gamma_2\Gamma_3\Gamma_4,$$
$$d_{23}=d_{32}=\frac{1}{3}\Gamma_1^2\Gamma_2+\frac{1}{3}\Gamma_1\Gamma_2^2-\frac{2}{3}\Gamma_1^2\Gamma_3+\frac{2}{3}\Gamma_1\Gamma_2\Gamma_3-\frac{2}{3}\Gamma_2^2\Gamma_3+\frac{1}{3}\Gamma_1^2\Gamma_4-$$
$$-\frac{4}{3}\Gamma_1\Gamma_2\Gamma_4+\frac{1}{3}\Gamma_2^2\Gamma_4+\frac{1}{3}\Gamma_1\Gamma_3\Gamma_4+\frac{1}{3}\Gamma_2\Gamma_3\Gamma_4,$$

$$d_{24}=d_{42}=\frac{1}{3}\Gamma_1^2\Gamma_2+\frac{1}{3}\Gamma_1\Gamma_2^2-\frac{2}{3}\Gamma_1^2\Gamma_3+\frac{2}{3}\Gamma_1\Gamma_2\Gamma_3-\frac{2}{3}\Gamma_2^2\Gamma_3+\frac{1}{3}\Gamma_1^2\Gamma_4-$$
$$-2\Gamma_1\Gamma_2\Gamma_4+\frac{1}{3}\Gamma_2^2\Gamma_4+\frac{2}{3}\Gamma_1\Gamma_3\Gamma_4+\frac{2}{3}\Gamma_2\Gamma_3\Gamma_4+\frac{1}{3}\Gamma_1\Gamma_4^2+\frac{1}{3}\Gamma_2\Gamma_4^2-\frac{2}{3}\Gamma_3\Gamma_4^2,$$
$$d_{34}=d_{43}=2\Gamma_1^2-\frac{4\Gamma_1\Gamma_2}{3}+2\Gamma_2^2-\frac{\Gamma_1^2\Gamma_2}{3\Gamma_3}-\frac{\Gamma_1\Gamma_2^2}{3\Gamma_3}-\frac{2\Gamma_1\Gamma_3}{3}-\frac{\Gamma_1^2\Gamma_3}{3\Gamma_2}-$$
$$-\frac{2\Gamma_2\Gamma_3}{3}-\frac{\Gamma_2^2\Gamma_3}{3\Gamma_1}-\frac{\Gamma_1^2\Gamma_2}{3\Gamma_4}-\frac{\Gamma_1\Gamma_2^2}{3\Gamma_4}-\frac{\Gamma_1^2\Gamma_3}{3\Gamma_4}+\frac{4\Gamma_1\Gamma_2\Gamma_3}{3\Gamma_4}-\frac{\Gamma_2^2\Gamma_3}{3\Gamma_4}-\frac{2\Gamma_1\Gamma_4}{3}-$$
$$-\frac{\Gamma_1^2\Gamma_4}{3\Gamma_2}-\frac{2\Gamma_2\Gamma_4}{3}-\frac{\Gamma_2^2\Gamma_4}{3\Gamma_1}-\frac{\Gamma_1^2\Gamma_4}{3\Gamma_3}+\frac{4\Gamma_1\Gamma_2\Gamma_4}{3\Gamma_3}-\frac{\Gamma_2^2\Gamma_4}{3\Gamma_3}-\frac{2\Gamma_3\Gamma_4}{3}+$$
$$+\frac{\Gamma_1\Gamma_3\Gamma_4}{\Gamma_2}+\frac{\Gamma_2\Gamma_3\Gamma_4}{\Gamma_1}.$$

According to Sylvester criterion this form will be positive definite if and only if 2nd and 4th minors are positive along with a product of 1st and 3d minor. After some simplifications these conditions can be transformed to (3.3.1). □

Without loss of generality we can put $\Gamma_1 = 1$. Then 3-parametric stable region is shown on Figure 3.2 and Figure 3.3.

Now consider degenerate case ($\mathbf{J} = 0$). To get $\sum_{i=1}^{4}\Gamma_i\mathbf{x}_i = 0$ we must have $\Gamma_1\mathbf{x}_1 = -\sum_{i=2}^{4}\Gamma_i\mathbf{x}_i$. For tetrahedron it is possible only if $\Gamma_2 = \Gamma_3 = \Gamma_4$. By the index rotation symmetry we will get $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4$. This means that only if $\Gamma_i = \Gamma_j$ we have degenerate tetrahedral relative equilibrium configuration.

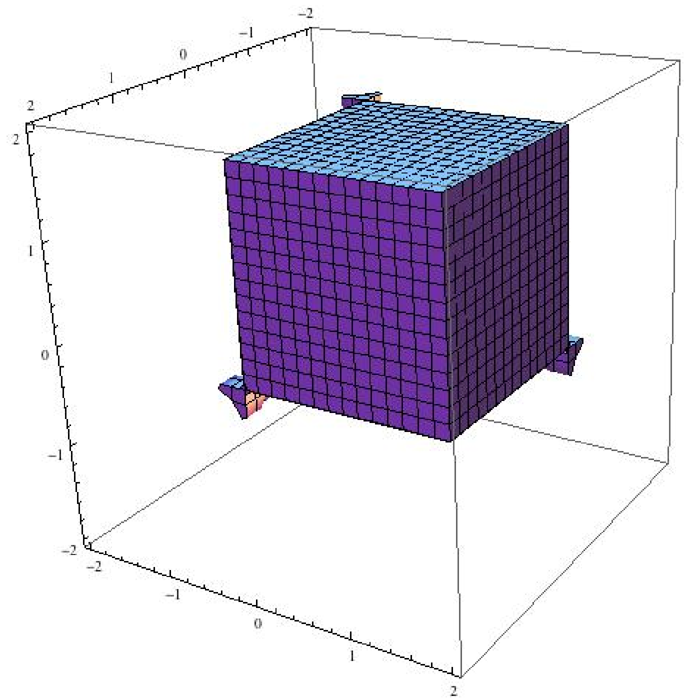


Figure 3.2: Stable region for tetrahedral configuration

Theorem 3.3.2. *Degenerate tetrahedral configuration ($\mathbf{J} = 0$) is a stable relative equilibrium configuration.*

Proof. Without loss of generality, assume

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4 = 1. \tag{3.3.3}$$

For this theorem we will use coordinate form of the energy-momentum method. Let us choose following coordinates for the vertices

$$\begin{aligned} z_1 &= \tfrac{1}{\sqrt{3}}, \quad \varphi_1 = \frac{\pi}{4}, \\ z_2 &= -\tfrac{1}{\sqrt{3}}, \quad \varphi_2 = \frac{3\pi}{4}, \\ z_3 &= -\tfrac{1}{\sqrt{3}}, \quad \varphi_3 = -\frac{\pi}{4}, \end{aligned}$$

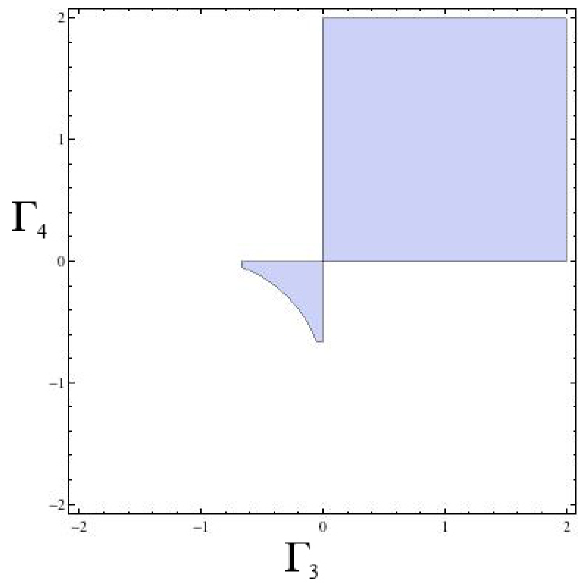


Figure 3.3: Stable region for tetrahedral configuration. Plane $\Gamma_1 = 1, \Gamma_2 = 2$

$$z_4 = \tfrac{1}{\sqrt{3}}, \quad \varphi_4 = -\frac{3\pi}{4}. \tag{3.3.4}$$

Then second variation of the energy-momentum functional will have following eigenvalues

$$-\frac{9}{4}, -\frac{21}{16}, -\frac{21}{16}, -\frac{9}{8}, -\frac{1}{2}, 0, 0, 0$$

Since $\dim \mathrm{so}(3) = 3$, we have to have 3 zero eigenvalues. All the others correspond to the transversal directions and, as we can see, they have the same sign. Thus by the Theorem 3.1.2 the configuration is stable. □

3.4 Stability of octahedral configurations

Similarly to tetrahedral case, octahedral configurations will be relative equilibria for any choice of $\Gamma_i, \ i = 1, .., 6$. Using techniques developed in previous section, we get

Theorem 3.4.1. *Non-degenerate octahedral configurations ($\mathbf{J} \neq 0$) are nonlinearly stable if*

$$\det[(d_{ij})_{i,j=1,\ldots,2}] > 0,$$
$$\det[(d_{ij})_{i,j=1,\ldots,4}] > 0,$$
$$\det[(d_{ij})_{i,j=1,\ldots,6}] > 0,$$
$$\det[(d_{ij})_{i,j=1,\ldots,8}] > 0,$$
$$\det[(d_{ij})_{i,j=1,\ldots,1}]\det[(d_{ij})_{i,j=1,\ldots,3}] > 0,$$
$$\det[(d_{ij})_{i,j=1,\ldots,1}]\det[(d_{ij})_{i,j=1,\ldots,5}] > 0,$$
$$\det[(d_{ij})_{i,j=1,\ldots,1}]\det[(d_{ij})_{i,j=1,\ldots,7}] > 0,$$

where

$$d_{11} = \frac{1}{2}\Gamma_1\Gamma_3\left(\Gamma_3\left(\Gamma_2 - 2\left(\Gamma_5 + \Gamma_6\right)\right) + \Gamma_1\left(\Gamma_4 - 2\left(\Gamma_5 + \Gamma_6\right)\right)\right),$$
$$d_{12} = d_{21} = \Gamma_1^2\Gamma_3\Gamma_6,$$
$$d_{13} = d_{31} = -\frac{1}{2}\Gamma_3^2\left(\Gamma_1 - \Gamma_2 + 2\left(\Gamma_5 + \Gamma_6\right)\right),$$
$$d_{14} = d_{41} = -\frac{1}{2}\Gamma_1^2\left(\Gamma_3 - \Gamma_4 + 2\left(\Gamma_5 + \Gamma_6\right)\right),$$
$$d_{15} = d_{51} = 2\Gamma_1\Gamma_3\left(\Gamma_1 + \Gamma_3\right),$$
$$d_{16} = d_{61} = 0,$$
$$d_{17} = d_{71} = -\frac{1}{2}\Gamma_1^2\left(\Gamma_3 - \Gamma_4 + 2\left(\Gamma_5 + \Gamma_6\right)\right),$$
$$d_{18} = d_{81} = 2\Gamma_1^2\Gamma_3,$$
$$d_{22} = \frac{1}{2}\Gamma_1\left(\left(\Gamma_2 - 2\left(\Gamma_3 + \Gamma_4\right)\right)\Gamma_5^2 - \Gamma_1\left(2\Gamma_5^2 + 3\Gamma_6\Gamma_5 + 2\Gamma_6\left(\Gamma_3 + \Gamma_4 + \Gamma_6\right)\right)\right),$$
$$d_{23} = d_{32} = 0,$$
$$d_{24} = d_{42} = 2\Gamma_1^2\Gamma_6,$$

$$d_{25} = d_{52} = -\frac{1}{2}\Gamma_1^2\left(2\Gamma_3 + 2\Gamma_4 - \Gamma_5 + \Gamma_6\right),$$

$$d_{26} = d_{62} = \frac{1}{2}\left(\Gamma_1 - \Gamma_2 + 2\left(\Gamma_3 + \Gamma_4\right)\right)\Gamma_5\Gamma_6,$$

$$d_{27} = d_{72} = 2\Gamma_1\left(\Gamma_1 - \Gamma_5\right)\Gamma_6,$$

$$d_{28} = d_{82} = -\frac{1}{2}\Gamma_1^2\left(2\Gamma_3 + 2\Gamma_4 - \Gamma_5 + \Gamma_6\right),$$

$$d_{33} = \frac{\Gamma_3^2\left(\Gamma_1^2 - 2\left(\Gamma_2 + \Gamma_5 + \Gamma_6\right)\Gamma_1 + \Gamma_2\left(\Gamma_2 - 2\left(\Gamma_5 + \Gamma_6\right)\right)\right)}{2\Gamma_1\Gamma_2},$$

$$d_{34} = d_{43} = 0,$$

$$d_{35} = d_{53} = 4\Gamma_3^2,$$

$$d_{36} = d_{63} = 0,$$

$$d_{37} = d_{73} = 0,$$

$$d_{38} = d_{83} = 0,$$

$$d_{44} = \frac{\Gamma_1^2\left(\Gamma_3^2 - 2\left(\Gamma_4 + \Gamma_5 + \Gamma_6\right)\Gamma_3 + \Gamma_4\left(\Gamma_4 - 2\left(\Gamma_5 + \Gamma_6\right)\right)\right)}{2\Gamma_3\Gamma_4},$$

$$d_{45} = d_{54} = 4\Gamma_1^2,$$

$$d_{46} = d_{64} = 0,$$

$$d_{47} = d_{74} = \frac{\Gamma_1^2\left(\Gamma_3^2 - 2\left(\Gamma_4 + \Gamma_5 + \Gamma_6\right)\Gamma_3 + \Gamma_4\left(\Gamma_4 - 2\left(\Gamma_5 + \Gamma_6\right)\right)\right)}{2\Gamma_3\Gamma_4},$$

$$d_{48} = d_{84} = 4\Gamma_1^2,$$

$$d_{55} = \frac{\left(\left(\Gamma_5 - \Gamma_6\right)^2 - 2\Gamma_3\left(\Gamma_5 + \Gamma_6\right) - 2\Gamma_4\left(\Gamma_5 + \Gamma_6\right)\right)\Gamma_1^2}{2\Gamma_5\Gamma_6} - \\ - \frac{2\Gamma_3^2\left(\Gamma_5 + \Gamma_6\right)\Gamma_1 + \Gamma_3^2\left(\left(\Gamma_5 - \Gamma_6\right)^2 - 2\Gamma_2\left(\Gamma_5 + \Gamma_6\right)\right)}{2\Gamma_5\Gamma_6},$$

$$d_{56} = d_{65} = 0,$$

$$d_{57} = d_{75} = 4\Gamma_1^2,$$

$$d_{58} = d_{85} = \frac{\Gamma_1^2\left(\left(\Gamma_5 - \Gamma_6\right)^2 - 2\Gamma_3\left(\Gamma_5 + \Gamma_6\right) - 2\Gamma_4\left(\Gamma_5 + \Gamma_6\right)\right)}{2\Gamma_5\Gamma_6},$$

$$d_{66} = \frac{\left(\Gamma_1^2 - 2\left(\Gamma_2 + \Gamma_3 + \Gamma_4\right)\Gamma_1 + \Gamma_2\left(\Gamma_2 - 2\left(\Gamma_3 + \Gamma_4\right)\right)\right)\Gamma_6^2}{2\Gamma_1\Gamma_2},$$

$$d_{67} = d_{76} = 4\Gamma_6^2,$$

$$d_{68} = d_{86} = 0,$$

$$d_{77} = \frac{\left(\left(\Gamma_3 - \Gamma_4\right)^2 - 2\left(\Gamma_3 + \Gamma_4\right)\Gamma_5\right)\Gamma_1^2 - 2\left(\Gamma_3 + \Gamma_4\right)\Gamma_6\Gamma_1^2}{2\Gamma_3\Gamma_4} + \\ + \frac{\left(\left(\Gamma_3 - \Gamma_4\right)^2 - 2\Gamma_1\left(\Gamma_3 + \Gamma_4\right) - 2\Gamma_2\left(\Gamma_3 + \Gamma_4\right)\right)\Gamma_6^2}{2\Gamma_3\Gamma_4},$$

$$d_{78} = d_{87} = 4\Gamma_1^2,$$

$$d_{88} = \frac{\Gamma_1^2\left(\left(\Gamma_5 - \Gamma_6\right)^2 - 2\Gamma_3\left(\Gamma_5 + \Gamma_6\right) - 2\Gamma_4\left(\Gamma_5 + \Gamma_6\right)\right)}{2\Gamma_5\Gamma_6}.$$

Proof. To prove the theorem we follow the steps we did for the tetrahedral configuration. Since $\mathbf{J} \neq 0$, thus $G = \mathrm{SO}(2)$ and tangent to the orbit space is $\mathfrak{g}_{\mu_e} = \mathrm{span}\{\mathbf{y}_o = (\mathbf{J} \times \mathbf{x}_1, \mathbf{J} \times \mathbf{x}_2, \mathbf{J} \times \mathbf{x}_3, \mathbf{J} \times \mathbf{x}_4, \mathbf{J} \times \mathbf{x}_5, \mathbf{J} \times \mathbf{x}_6)\}$. To find $\mathfrak{C}_1$ we choose two linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$ in each copy of S^2. Lets also choose these vectors to be non-coplanar with any two from $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6, \mathbf{J}$. Consider basis for $T_{\mathbf{x}_e}P$:

$$\mathbf{e}^{(1)} = (\mathbf{x}_1 \times \mathbf{v}_1, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}),$$

$$\mathbf{e}^{(2)} = (\mathbf{0}, \mathbf{x}_2 \times \mathbf{v}_1, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}),$$

$$\mathbf{e}^{(3)} = (\mathbf{0}, \mathbf{0}, \mathbf{x}_3 \times \mathbf{v}_1, \mathbf{0}, \mathbf{0}, \mathbf{0}),$$

$$\mathbf{e}^{(4)} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{x}_4 \times \mathbf{v}_1, \mathbf{0}, \mathbf{0}),$$

$$\mathbf{e}^{(5)} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{x}_5 \times \mathbf{v}_1, \mathbf{0}),$$

$$\mathbf{e}^{(6)} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{x}_6 \times \mathbf{v}_1),$$

$$\mathbf{e}^{(7)} = (\mathbf{x}_1 \times \mathbf{v}_2, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}),$$

$$\mathbf{e}^{(8)} = (\mathbf{0}, \mathbf{x}_2 \times \mathbf{v}_2, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}),$$

$$\mathbf{e}^{(9)} = (\mathbf{0}, \mathbf{0}, \mathbf{x}_3 \times \mathbf{v}_2, \mathbf{0}, \mathbf{0}, \mathbf{0}),$$

$$\mathbf{e}^{(10)} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{x}_4 \times \mathbf{v}_2, \mathbf{0}, \mathbf{0}),$$

$$\mathbf{e}^{(11)} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{x}_5 \times \mathbf{v}_2, \mathbf{0}),$$

$$\mathbf{e}^{(12)} = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{x}_6 \times \mathbf{v}_2),$$

Let $\alpha_i, i = 1, .., 6 \;\; \beta_i, i = 1, .., 6$ be the solutions of

$$\begin{cases} \sum\limits_{i=1}^{6} \Gamma_i \alpha_i \mathbf{x}_i = \mathbf{v}_1, \text{ or } \sum\limits_{i=1}^{6} \Gamma_i \alpha_i \mathbf{x}_i = 0, \\ \sum\limits_{i=1}^{6} \Gamma_i \beta_i \mathbf{x}_i = \mathbf{v}_2, \text{ or } \sum\limits_{i=1}^{6} \Gamma_i \beta_i \mathbf{x}_i = 0. \end{cases} \tag{3.4.1}$$

Then $\mathbf{y} = \sum\limits_{i=1}^{6} \alpha_i \mathbf{e}^{(i)} + \sum\limits_{i=1}^{4} \beta_i \mathbf{e}^{(i+4)}$ will belong to $\ker D\mathbf{J}(\mathbf{x}_e)$. From $\mathbf{J} \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ we have

$$\mathbf{J} \notin K = \operatorname{span}\{\sum_{i=1}^{6} \alpha_i \mathbf{e}^{(i)} + \sum_{i=1}^{6} \beta_i \mathbf{e}^{(i+6)} | \alpha_i, \beta_i \text{ - solutions of (3.4.1)}\}.$$

From (3.1.24) we have that dimension of $\ker D\mathbf{J}(\mathbf{x}_e)$ is $2N - 3 = 12 - 3 = 9$. Dimension of $\mathfrak{g}_{\mu_e}$ is 1, thus $\dim(\mathfrak{C}_1) = 9 - 1 = 8$. Since every equation in (3.4.1) has $N - 2 = 4$ linearly independent solution and vectors $\mathbf{v}_1$, $\mathbf{v}_2$ are linearly independent we obtain $K = \mathfrak{C}_1$.

Choose the coordinates of the vertices

$$\begin{aligned} \mathbf{x}_1 &= (1, 0, 0), \\ \mathbf{x}_2 &= (-1, 0, 0), \\ \mathbf{x}_3 &= (0, 1, 0), \\ \mathbf{x}_4 &= (0, -1, 0), \\ \mathbf{x}_5 &= (0, 0, 1), \\ \mathbf{x}_6 &= (0, 0, -1). \end{aligned}$$

Then from (3.1.22) we have

$$\begin{aligned}
k_1 &= \frac{\Gamma_1\Gamma_2 - 2\Gamma_1^2}{16\pi},\ k_2 = \frac{\Gamma_1\Gamma_2 - 2\Gamma_2^2}{16\pi},\\
k_3 &= \frac{\Gamma_3\Gamma_4 - 2\Gamma_3^2}{16\pi},\ k_4 = \frac{\Gamma_3\Gamma_4 - 2\Gamma_4^2}{16\pi},\\
k_5 &= \frac{\Gamma_5\Gamma_6 - 2\Gamma_5^2}{16\pi},\ k_6 = \frac{\Gamma_5\Gamma_6 - 2\Gamma_6^2}{16\pi},\\
\boldsymbol{\xi} &= \left(\frac{\Gamma_1 - \Gamma_2}{4\pi}, \frac{\Gamma_3 - \Gamma_4}{4\pi}, \frac{\Gamma_5 - \Gamma_6}{4\pi}\right).
\end{aligned}$$

Let $\mathbf{v}_1 = \Gamma_1\mathbf{x}_1 + \Gamma_3\mathbf{x}_3 + \Gamma_5\mathbf{x}_5$ and $\mathbf{v}_2 = \Gamma_1\mathbf{x}_1 + \Gamma_3\mathbf{x}_3 + \Gamma_6\mathbf{x}_6$. Since none of the $\Gamma_i,\ i = 1, .., 6$ is zero, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6, \mathbf{J}$ are not in $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. One of the simplest solutions of the system (3.4.1) are

$$\begin{aligned}
&\alpha_1^{(1)} = 1,\ \alpha_3^{(1)} = 1,\ \alpha_5^{(1)} = 1, \alpha_j^{(1)} = 0,\ \beta_i^{(1)} = 0,\ i = 1, .., 6,\ j = 2, 4, 6.\\
&\beta_1^{(2)} = 1,\ \beta_3^{(2)} = 1,\ \beta_6^{(2)} = 1, \beta_j^{(2)} = 0,\ \alpha_i^{(2)} = 0,\ i = 1, .., 6,\ j = 2, 4, 5.\\
&\alpha_1^{(3)} = \frac{1}{\Gamma_1},\ \alpha_2^{(3)} = \frac{1}{\Gamma_2},\ \alpha_j^{(3)} = 0,\ \beta_i^{(3)} = 0,\ i = 1, .., 6,\ j = 3, .., 6.\\
&\alpha_3^{(4)} = \frac{1}{\Gamma_3},\ \alpha_4^{(4)} = \frac{1}{\Gamma_4},\ \alpha_j^{(4)} = 0,\ \beta_i^{(4)} = 0,\ i = 1, .., 6,\ j = 1, 2, 5, 6.\\
&\alpha_5^{(5)} = \frac{1}{\Gamma_5},\ \alpha_6^{(5)} = \frac{1}{\Gamma_6},\ \alpha_j^{(5)} = 0,\ \beta_i^{(5)} = 0,\ i = 1, .., 6,\ j = 1, .., 4.\\
&\beta_1^{(6)} = \frac{1}{\Gamma_1},\ \beta_2^{(6)} = \frac{1}{\Gamma_2},\ \beta_j^{(6)} = 0,\ \alpha_i^{(6)} = 0,\ i = 1, .., 6,\ j = 3, .., 6.\\
&\beta_3^{(7)} = \frac{1}{\Gamma_3},\ \beta_4^{(7)} = \frac{1}{\Gamma_4},\ \beta_j^{(7)} = 0,\ \alpha_i^{(7)} = 0,\ i = 1, .., 6,\ j = 1, 2, 5, 6.\\
&\beta_5^{(8)} = \frac{1}{\Gamma_5},\ \beta_6^{(8)} = \frac{1}{\Gamma_6},\ \beta_j^{(8)} = 0,\ \alpha_i^{(8)} = 0,\ i = 1, .., 6,\ j = 1, .., 4.
\end{aligned}$$

Thus basis of $\mathfrak{C}_1$ is

$$
\begin{aligned}
\boldsymbol{\epsilon}^{(1)} &= \mathbf{e}^{(1)} + \mathbf{e}^{(3)} + \mathbf{e}^{(5)},\\
\boldsymbol{\epsilon}^{(2)} &= \mathbf{e}^{(7)} + \mathbf{e}^{(9)} + \mathbf{e}^{(12)},\\
\boldsymbol{\epsilon}^{(3)} &= \frac{1}{\Gamma_1}\mathbf{e}^{(1)} + \frac{1}{\Gamma_2}\mathbf{e}^{(2)},\\
\boldsymbol{\epsilon}^{(4)} &= \frac{1}{\Gamma_3}\mathbf{e}^{(3)} + \frac{1}{\Gamma_4}\mathbf{e}^{(4)},\\
\boldsymbol{\epsilon}^{(5)} &= \frac{1}{\Gamma_5}\mathbf{e}^{(5)} + \frac{1}{\Gamma_6}\mathbf{e}^{(6)},\\
\boldsymbol{\epsilon}^{(6)} &= \frac{1}{\Gamma_1}\mathbf{e}^{(7)} + \frac{1}{\Gamma_2}\mathbf{e}^{(8)},\\
\boldsymbol{\epsilon}^{(7)} &= \frac{1}{\Gamma_3}\mathbf{e}^{(9)} + \frac{1}{\Gamma_4}\mathbf{e}^{(10)},\\
\boldsymbol{\epsilon}^{(8)} &= \frac{1}{\Gamma_5}\mathbf{e}^{(11)} + \frac{1}{\Gamma_6}\mathbf{e}^{(12)}.
\end{aligned}
$$

Second variation of energy-momentum along $\boldsymbol{\epsilon}^{(j)},\ j = 1, .., 8$ is

$$D^2 H_{\mu_e}|_{\boldsymbol{\epsilon}^{(j)}, j=1,..,8} = (d_{ij})_{i=1,..,8, j=1,..,8}.$$

And the second variation is definite if all of the even minors are positive as well as products of odd minors. □

In order to get some understanding of the stability region, we put $\Gamma_i = 1,\ i = 1, \ldots, 3$. Then the 3-parametric projection of stable region is shown on Figure 3.4.

Since octahedron has four vertices in one plane and other two on perpendicular line to the plane, the only possibility for degenerate relative equilibria is the case $\Gamma_i = \Gamma_j,\ i, j = 1, .., 6$.

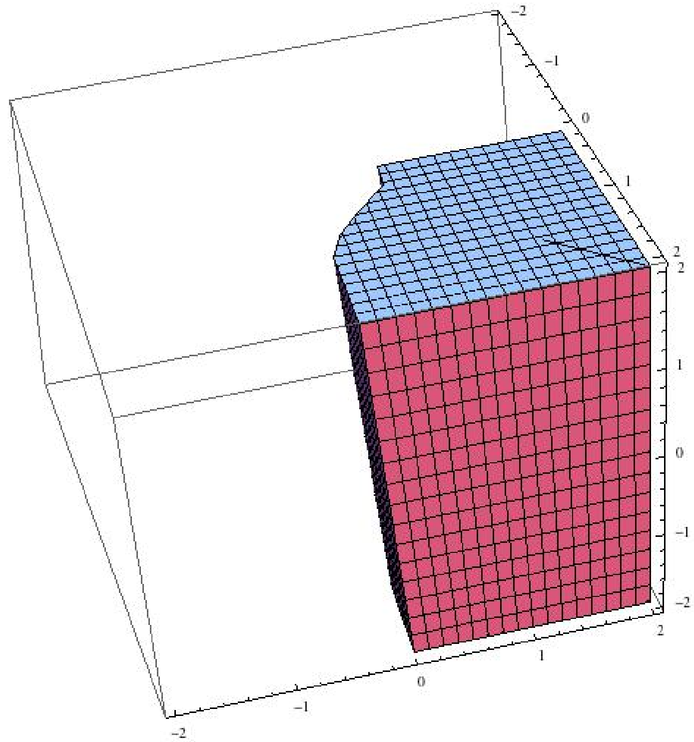


Figure 3.4: Stable region for octahedral configuration

Theorem 3.4.2. *Degenerate octahedral configuration ($\mathbf{J} = 0$) is a stable relative equilibrium configuration.*

Proof. Here we repeat the argument we used for tetrahedral configuration. Without loss of generality, assume

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4 = \Gamma_5 = \Gamma_6 = 1. \tag{3.4.2}$$

Using the coordinate form of the energy-momentum method with coordinates

$$
\begin{aligned}
z_1 &= \tfrac{1}{\sqrt{2}}, \quad \varphi_1 = 0, \\
z_2 &= \tfrac{1}{\sqrt{2}}, \quad \varphi_2 = \frac{2\pi}{3}, \\
z_3 &= \tfrac{1}{\sqrt{2}}, \quad \varphi_3 = -\frac{2\pi}{3},
\end{aligned}
$$

$$z_4 = -\tfrac{1}{\sqrt{2}}, \quad \varphi_4 = \frac{\pi}{3}.$$
$$z_5 = -\tfrac{1}{\sqrt{2}}, \quad \varphi_5 = -\frac{\pi}{3},$$
$$z_6 = -\tfrac{1}{\sqrt{2}}, \quad \varphi_6 = \pi,$$

we get second variation of the energy-momentum functional with following eigenvalues

$$-2, -\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}, 0, 0, 0$$

Since $\dim \mathrm{so}(3) = 3$, we have to have 3 zero eigenvalues. All the others correspond to the transversal directions and since they have the same sign, by the Theorem 3.1.2 the configuration is stable. □

3.5 Stability of cubic configurations

As we know from the previous chapter, configuration matrix for the cube has five dimensional null space. This means that in order to do the general stability analysis we have to introduce 5 parameters and then study 8*2-4=12 dimensional second variation matrix. To simplify computations and to be able to visualize the regions of stability in this section we will study stability of superposition of the axis-symmetric cubic configurations which are represented on Figure 3.5.

Theorem 3.5.1. *Cubic configurations are nonlinearly stable if*

$$\left(6\Gamma^2 - \Gamma_\alpha \left(\Gamma_\alpha + 3\Gamma_\beta\right)\right) > 0,$$
$$\mathbf{B}_{1,1}\mathbf{B}_{2i-1,2i-1} > 0,$$
$$\mathbf{B}_{2i,2i} > 0,\ i = 1, .., 5.$$

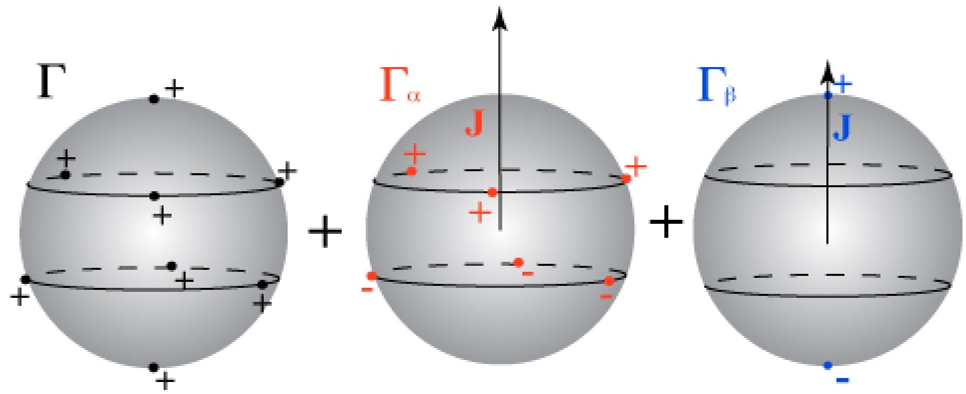


Figure 3.5: Superposition of axis-symmetric cubic configurations

where $\mathbf{B}_{i,i}$ *is* i*th principal minor of* $\mathbf{B}$*, which is given in Appendix B.*

Proof. Since the configurations are axis-symmetric, it is convenient to use cylindrical coordinates aligned with vector $\mathbf{J}$. If we chose coordinates of the vertices of the cube in the form (2.3.3) and then use symmetry induced basis for the space $\mathfrak{C}_1$ we get following matrix of second variation

$$D^2 H_{\mu_e}|_{\mathfrak{C}_1} = \begin{pmatrix} -\frac{64}{3}\left(\Gamma^2-\Gamma_\alpha^2\right) & 0 & 0 \\ 0 & -\frac{27}{8}\left(\Gamma^2-\Gamma_\alpha^2\right)\left(6\Gamma^2-\Gamma_\alpha\left(\Gamma_\alpha+3\Gamma_\beta\right)\right) & 0 \\ 0 & 0 & \mathbf{B} \end{pmatrix}, \tag{3.5.1}$$

where matrix $\mathbf{B}$ is a 10x10 matrix with components given in Appendix B.

Since matrix $D^2 H_{\mu_e}|_{\mathfrak{C}_1}$ has to be definite, using Sylvester criterion we will get the conditions of the theorem. □

Visualizations of these conditions are given on the plots Figure 3.6 and Figure 3.7. The second plot is a plane $\Gamma = 1$. All the other planes $\Gamma = z$ are just rescaled versions of each other.

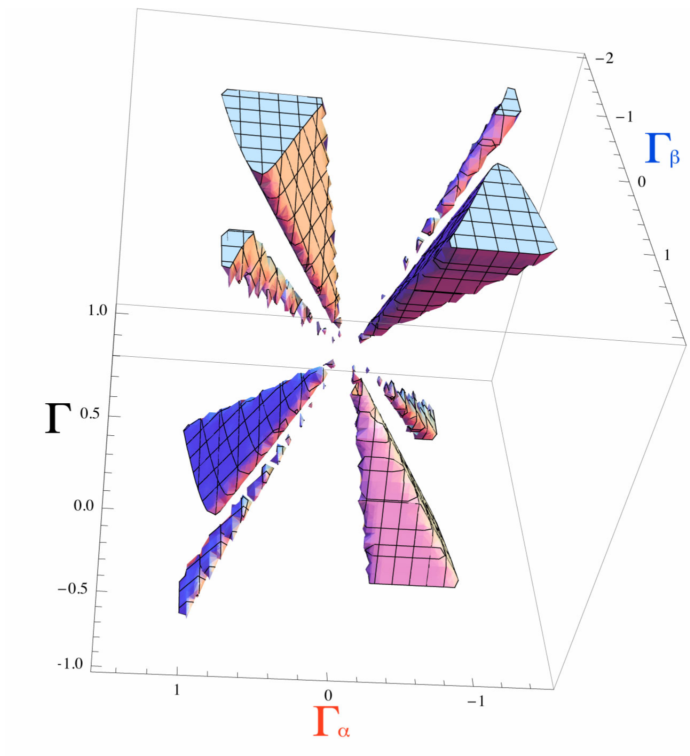


Figure 3.6: Stability region for cubic configurations

As we can see from the plots, configurations $(\Gamma, \Gamma_\alpha, \Gamma_\beta) \in \{(1,0,0),(0,1,0)\}$ are not stable. We can even prove that they are unstable.

Theorem 3.5.2. *Relative equilibrium configurations* $(\Gamma, \Gamma_\alpha, \Gamma_\beta) \in \{(1,0,0),(0,1,0)\}$ *are linearly unstable configurations.*

Proof. Matrix of linearized system can be obtained from the second variation of Hamiltonian by multiplying the second variation by inverse of symplectic form (3.1.30). Eigenvalues of linearized system for configuration $(\Gamma, \Gamma_\alpha, \Gamma_\beta) = (1,0,0)$ are

$$i\sqrt{3}, i\sqrt{3}, i\sqrt{3}, -i\sqrt{3}, -i\sqrt{3}, -i\sqrt{3}, -\frac{\sqrt{15}}{4}, -\frac{\sqrt{15}}{4}, \frac{\sqrt{15}}{4}, \frac{\sqrt{15}}{4}, 0, 0, 0, 0, 0, 0$$

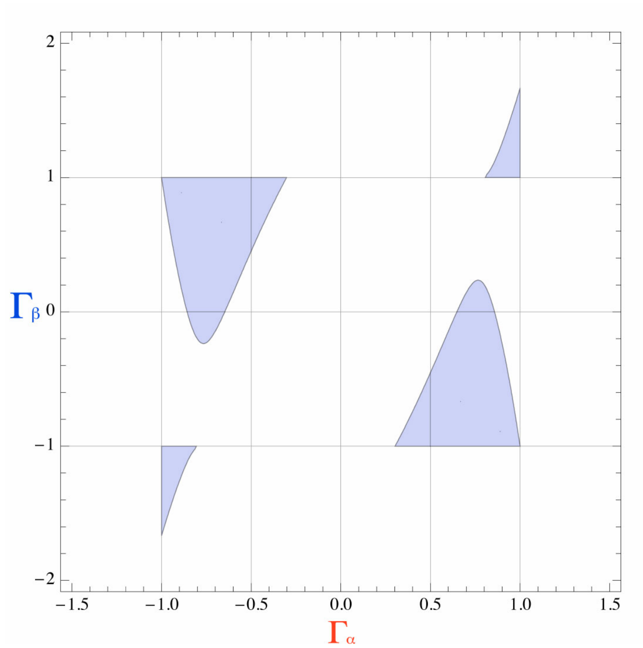


Figure 3.7: Stability region for cubic configurations. Plane $\Gamma = 1$

And for configuration $(\Gamma, \Gamma_\alpha, \Gamma_\beta) = (0, 1, 0)$

$$-\sqrt{\frac{65}{64}+\frac{\sqrt{145}}{32}}, -\sqrt{\frac{65}{64}+\frac{\sqrt{145}}{32}}, \sqrt{\frac{65}{64}+\frac{\sqrt{145}}{32}}, \sqrt{\frac{65}{64}+\frac{\sqrt{145}}{32}}, -\frac{1}{8}\sqrt{65-2\sqrt{145}},$$
$$-\frac{1}{8}\sqrt{65-2\sqrt{145}}, \frac{1}{8}\sqrt{65-2\sqrt{145}}, \frac{1}{8}\sqrt{65-2\sqrt{145}}, \frac{i}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, 0, 0$$

As we can see both of the configurations have eigenvalues with positive real parts. Thus the configurations are unstable. □

Notice that configuration $(\Gamma, \Gamma_\alpha, \Gamma_\beta) = (0, 0, 1)$ is a vortex pair and, as it was proven above, it is stable. But this stability is not captured by the energy-momentum method.

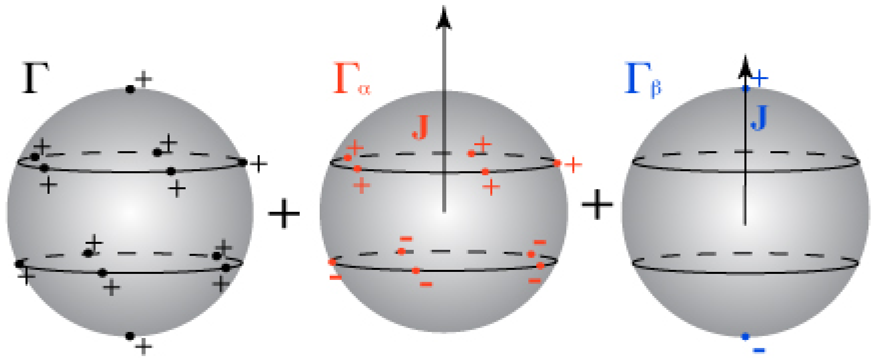


Figure 3.8: Superposition of axis-symmetric icosahedral configurations

3.6 Stability of icosahedral configurations

From the previous chapter we know that configuration matrix of icosahedral configuration has the seven dimensional null space. Thus general stability analysis will be in seven dimensional space. As in the cubic case, to simplify computations and to be able to visualize the regions of stability we will study stability of superposition of the axis-symmetric icosahedral configurations which are represented on Fig.3.8.

Theorem 3.6.1. *Icosahedral configurations are nonlinearly stable if*

$$
\begin{aligned}
\left(10\Gamma^2 - \Gamma_\alpha\left(4\Gamma_\alpha + \sqrt{5}\Gamma_\beta\right)\right) > 0, \\
-8\left(\Gamma^2 - \Gamma_\alpha^2\right)\mathbf{C}_{1,1} > 0, \\
-8\left(\Gamma^2 - \Gamma_\alpha^2\right)\mathbf{D}_{1,1} > 0, \\
\mathbf{C}_{1,1}\mathbf{C}_{2i-1,2i-1} > 0, \\
\mathbf{C}_{2i,2i} > 0,\ i = 1, .., 4, \\
\mathbf{D}_{1,1}\mathbf{D}_{2i-1,2i-1} > 0, \\
\mathbf{D}_{2i,2i} > 0,\ i = 1, .., 5.
\end{aligned}
$$

where $\mathbf{C}_{i,i}$ is ith principal minor of matrix $\mathbf{C}$. Matrices $\mathbf{C}$ and $\mathbf{D}$ depend on the intensities Γ_i and their components can be found in Appendix B.

Proof. Since the configurations are axis-symmetric, we use cylindrical coordinates aligned with vector $\mathbf{J}$. If we chose coordinates of the vertices of the icosahedron in the form (2.3.4) and then use symmetry induced basis for the space $\mathfrak{C}_1$ we get following matrix of second variation

$$D^2 H_{\mu_e}|_{\mathfrak{C}_1} = \begin{pmatrix} -8\left(\Gamma^2-\Gamma_\alpha^2\right) & 0 & 0 & 0 \\ 0 & -\frac{25}{2}\left(\Gamma^2-\Gamma_\alpha^2\right)\left(10\Gamma^2-\Gamma_\alpha\left(4\Gamma_\alpha+\sqrt{5}\Gamma_\beta\right)\right) & 0 & 0 \\ 0 & 0 & \mathbf{C} & 0 \\ 0 & 0 & 0 & \mathbf{D} \end{pmatrix}, \tag{3.6.1}$$

where $\mathbf{C}$ is a 8×8 matrix and $\mathbf{D}$ is a 10×10 matrix[6].

Since matrix $D^2 H_{\mu_e}|_{\mathfrak{C}_1}$ has to be definite, using Sylvester criterion we get the conditions of the theorem. □

Visualizations of these conditions are given on the plots Fig.3.9 and Fig.3.10. The second plot is a plane $\Gamma = 1$. Regions of stability are self-similar in each parallel section of the region. This agrees with the observation that the problem allows linear rescaling of Γ's.

Notice, that in contrast to the cubic case, icosahedral configuration $(\Gamma, \Gamma_\alpha, \Gamma_\beta) = (1, 0, 0)$ is a stable relative equilibrium configuration. It has growing with Γ region of stability and this can be used to stabilize relative equilibrium, since if we increase Γ sufficiently, we will get inside of the stability region.

[6]See Appendix B for details.

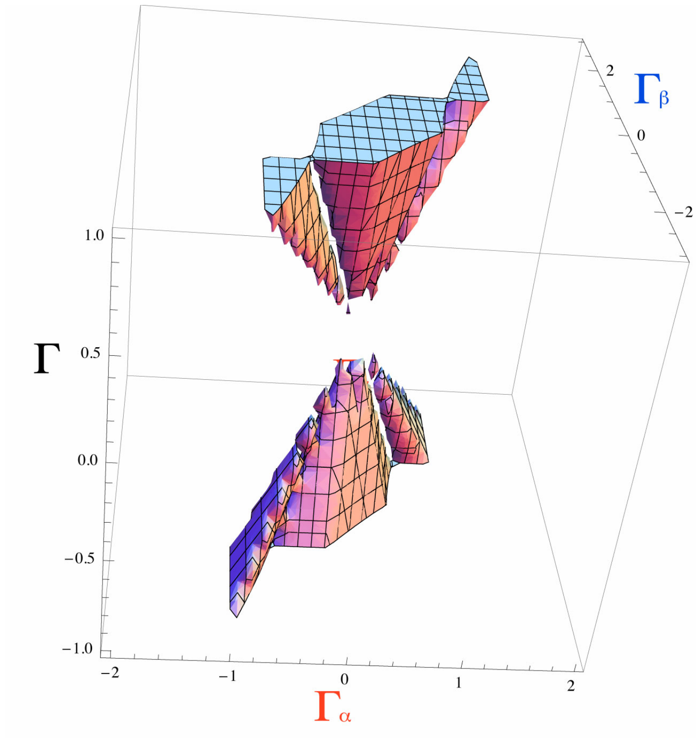


Figure 3.9: Stability region for icosahedral configuration

As we noted in the previous section, configuration $(\Gamma, \Gamma_\alpha, \Gamma_\beta) = (0, 0, 1)$ is a polar vortex pair and is a stable configuration.

Theorem 3.6.2. *Relative equilibrium configuration* $(\Gamma, \Gamma_\alpha, \Gamma_\beta) = (0, 1, 0)$ *is linearly unstable.*

Proof. As before, if we multiply matrix of second variation of the Hamiltonian by the inverse of the symplectic form then we can find the eigenvalues of linearized system

$$-1.98083, 1.98083, 1.98083, -1.98083, 1.86933i, -1.86933i, 1.86933i, -1.86933i,$$
$$1.43418, -1.43418, -1.43418, 1.43418, 1.32288i, -1.32288i,$$
$$-0.513637, 0.513637, 0.513637, -0.513637, 0, 0.$$

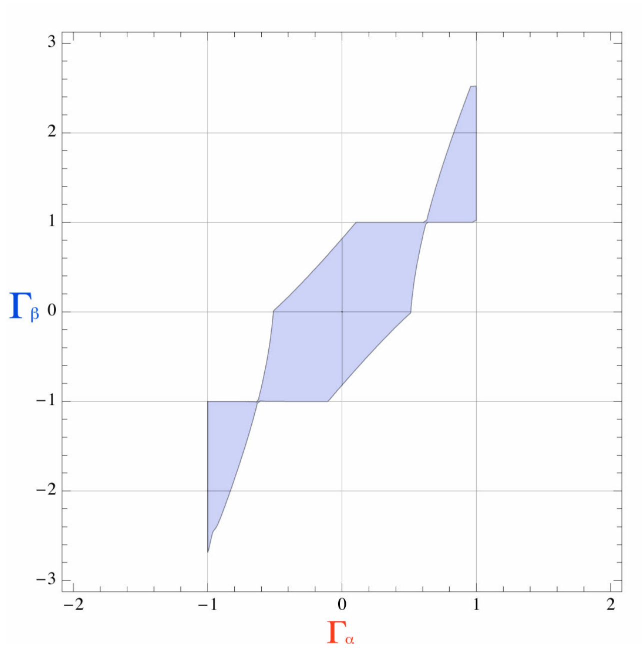


Figure 3.10: Stability region for icosahedral configuration. Plane $\Gamma = 1$

As we can see there are six of them with positive real part. Thus the configuration is linearly unstable. □

3.7 Stability of dodecahedral configurations

As it was shown in previous chapter, configuration matrix for the dodecahedron has 4 dimensional null space. In this section we will study stability of axis-symmetric linear superposition of two vectors of intensities $\Gamma = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ and $\Gamma_\alpha = (1+\phi, 1+\phi, 1+\phi, 1+\phi, 1+\phi, \phi, \phi, \phi, \phi, \phi, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)$ (see Figure 3.11).

Theorem 3.7.1. *Dodecahedral configurations Γ, Γ_α and $\Gamma + \Gamma_\alpha$ are unstable.*

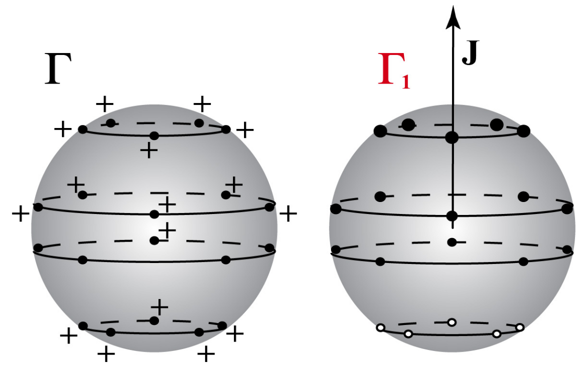


Figure 3.11: Superposition of axis-symmetric icosahedral configurations

Proof. As in previous section, by using cylindrical coordinates and symmetrically adapted basis we can find that the Eigenvalues of linearized system for the configuration Γ are

$$0. + 9.33742i, 0. - 9.33742i, 0. + 9.33742i, 0. - 9.33742i,$$
$$0. + 9.33742i, 0. - 9.33742i, 0. + 9.33742i, 0. - 9.33742i, 0. + 9.32281i,$$
$$0. - 9.32281i, 0. + 9.32281i, 0. - 9.32281i, 0. + 9.32281i,$$
$$0. - 9.32281i, 0. + 9.32281i, 0. - 9.32281i, 0. + 9.32281i,$$
$$0. - 9.32281i, -6.25625, 6.25625, -6.25625, 6.25625, 6.25625, -6.25625,$$
$$-5.44797, 5.44797, -5.44797, 5.44797, 5.44797, 5.44797, 5.44797,$$
$$-5.44797, -5.44797, -5.44797, 0, 0, 0, 0, 0, 0$$

Thus the configuration is unstable. The eigenvalues for Γ_α

$$0. + 23.0426i, 0. - 23.0426i, 0. + 18.832i, 0. - 18.832i, 0. + 16.4675i,$$
$$0. - 16.4675i, 0. + 14.8255i, 0. - 14.8255i, 0. + 13.9323i, 0. - 13.9323i,$$
$$-12.3428, 12.3428, 0. + 11.1227i, 0. - 11.1227i, 9.34178 + 0.63591i,$$

$$9.34178 - 0.63591i, -9.34178 + 0.63591i, -9.34178 - 0.63591i,$$
$$0. + 6.90988i, 0. - 6.90988i, 6.06939 + 0.59737i, 6.06939 - 0.59737i,$$
$$-6.06939 + 0.59737i, -6.06939 - 0.59737i, -3.38402 + .546296i,$$
$$-3.38402 - 0.546296i, 3.38402 + 0.546296i, 3.38402 - 0.546296i,$$
$$2.14816, -2.14816, -2.14816, -2.14816, -2.14816,$$
$$-2.14816, 2.14816, 2.14816, 2.14816, 2.14816, 0, 0$$

The configuration is unstable as well. And for the configuration $\Gamma_\alpha + \Gamma$

$$0. + 32.4149i, 0. - 32.4149i, 0. + 28.4896i, 0. - 28.4896i,$$
$$0. + 26.314i, 0. - 26.314i, 0. + 23.7701i, 0. - 23.7701i,$$
$$0. + 20.9321i, 0. - 20.9321i, -18.2794, 18.2794, 0. + 17.5358i,$$
$$0. - 17.5358i, 15.7541 + 0.584415i,$$
$$15.7541 - 0.584415i, -15.7541 + 0.584415i, -15.7541 - 0.584415i, 0. + 14.8255i,$$
$$0. - 14.8255i, 0. + 14.3331i, 0. - 14.3331i, 0. + 12.5094i,$$
$$0. - 12.5094i, -12.1304 + 0.891611i,$$
$$-12.1304 - 0.891611i, 12.1304 + 0.891611i,$$
$$12.1304 - 0.891611i, 0. + 10.232i, 0. - 10.232i,$$
$$-9.79623 + 1.41522i, -9.79623 - 1.41522i, 9.79623 + 1.41522i,$$
$$9.79623 - 1.41522i, -8.47321, 8.47321, 0. + 0.660075i, 0. - 0.660075i, 0, 0$$

Thus the configurations are unstable. □

Part II

Point singularities on a plane

Chapter 4

Introduction

We start the the second part by introducing the model of complex point singularity. We derive equations of motion by doing linear superposition of velocity vector fields. Then we consider symmetries of the system.

4.1 Equations of motion

Consider the vector field at $z = 0$ governed by the complex dynamical system:

$$\dot{z}^* = \frac{\Gamma}{2\pi i}\frac{1}{z}, \quad z(t) \in \mathbb{C}, \quad \Gamma \in \mathbb{C}, \quad t \in \mathbb{R} > 0, \tag{4.1.1}$$

where z^* denotes the complex conjugate of $z(t)$. Letting $z(t) = r(t)\exp(i\theta(t))$, $\Gamma = \Gamma_r + i\Gamma_i$, gives:

$$\dot{r} = \frac{\Gamma_i}{2\pi r}, \tag{4.1.2}$$

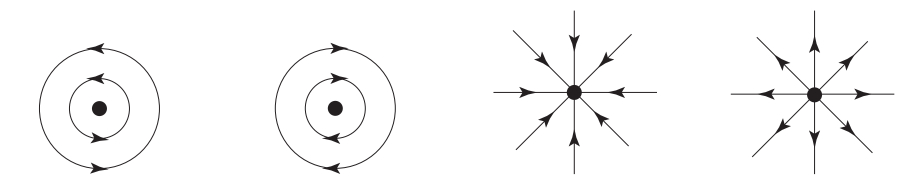
$$\dot{\theta} = \frac{\Gamma_r}{2\pi r^2}, \tag{4.1.3}$$

from which it is easy to see that:

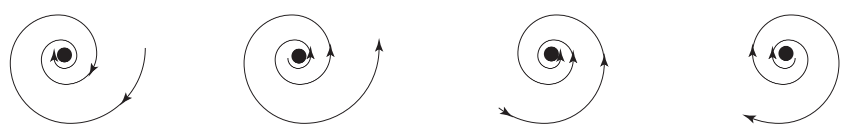
$$r(t) = \sqrt{\left(\frac{\Gamma_i}{2\pi}\right) t + r^2(0)}, \tag{4.1.4}$$

$$\theta(t) = \begin{cases} \left(\frac{\Gamma_r}{\Gamma_i}\right) \ln\left(\left(\frac{\Gamma_r}{\Gamma_i}\right) t + r^2(0)\right) & \text{if } \Gamma_i \neq 0 \\ \\ \frac{\Gamma_r t}{2\pi r^2(0)} + \theta(0) & \text{if } \Gamma_i = 0. \end{cases} \tag{4.1.5}$$

When $\Gamma_r \neq 0$, $\Gamma_i = 0$, the field is that of a classical point-vortex (figure 4.1(a),(b)); when $\Gamma_r = 0$, $\Gamma_i \neq 0$ it is a source ($\Gamma_i > 0$) or sink ($\Gamma_i < 0$) (figure 4.1(c),(d)), while when $\Gamma_r \neq 0$, $\Gamma_i \neq 0$, it is a spiral-source or sink (figure 4.1(e)-(h)).



(a) $\Gamma_r > 0, \Gamma_i = 0$ (b) $\Gamma_r < 0, \Gamma_i = 0$ (c) $\Gamma_r = 0, \Gamma_i < 0$ (d) $\Gamma_r = 0, \Gamma_i > 0$



(e) $\Gamma_r < 0, \Gamma_i < 0$ (f) $\Gamma_r > 0, \Gamma_i > 0$ (g) $\Gamma_r > 0, \Gamma_i < 0$ (h) $\Gamma_r < 0, \Gamma_i > 0$

Figure 4.1: All possible flowfields at the singular point $z = 0$ associated with the dynamical system (4.1.1).

A collection of N of these point singularities, each located at $z = z_\beta(t)$, $\beta = 1, ..., N$, by linear superposition, produces the field:

$$\dot{z}^* = \frac{1}{2\pi i} \sum_{\beta=1}^{N} \frac{\Gamma_\beta}{z - z_\beta}; \quad z(t) \equiv x(t) + iy(t) \in \mathbb{C}, \quad \Gamma_\beta \in \mathbb{C}. \tag{4.1.6}$$

Then, if we advect each by the velocity field generated by all the others[1], we arrive at the complex dynamical system:

$$\dot{z}^*_\alpha = \frac{1}{2\pi i}\sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta}, \quad z_\alpha(t) \equiv x_\alpha(t) + i y_\alpha(t) \in \mathbb{C}, \quad \Gamma_\beta \in \mathbb{C}, \tag{4.1.7}$$

where $'$ indicates that $\beta \neq \alpha$.

4.2 Symmetries of the system

It is easy to see that the system doesn't have a canonical Hamiltonian structure. Indeed, for example is case of 2 point singularities the equations of motion are

$$\begin{aligned}\dot{z}^*_1 &= \frac{1}{2\pi i}\frac{\Gamma_2}{z_1 - z_2},\\ \dot{z}^*_2 &= \frac{1}{2\pi i}\frac{\Gamma_1}{z_2 - z_1},\end{aligned}$$

or in cartesian coordinates

$$\begin{aligned}\dot{x}_1 &= \frac{1}{2\pi}\frac{\Gamma^i_2(x_1 - x_2) + \Gamma^r_2(y_1 - y_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2},\\ \dot{y}_1 &= \frac{1}{2\pi}\frac{\Gamma^r_2(x_1 - x_2) - \Gamma^r_2(y_1 - y_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2},\\ \dot{x}_2 &= \frac{1}{2\pi}\frac{\Gamma^i_1(x_2 - x_1) + \Gamma^r_1(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2},\\ \dot{y}_2 &= \frac{1}{2\pi}\frac{\Gamma^r_1(x_2 - x_1) - \Gamma^r_1(y_2 - y_1)}{(x_2 - x_1)^2 + (y_2 - y_1)^2},\end{aligned}$$

[1]One might characterize this dynamical assumption by saying that each singularity 'goes with the flow'.

where $\Gamma_j = \Gamma_j^r + i\Gamma_j^i$, $j = 1, 2$. Then in order to have canonical Hamiltonian system we must have

$$\dot{x}_j = C\frac{\partial H}{\partial y_j},$$
$$\dot{y}_j = -C\frac{\partial H}{\partial x_j},$$

where $C \in \{1, -1\}$. Thus

$$C\frac{\partial H}{\partial y_1} = \frac{1}{2\pi}\frac{\Gamma_2^i(x_1 - x_2) + \Gamma_2^r(y_1 - y_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2}, \Rightarrow$$
$$H = C\frac{\Gamma_2^r}{4\pi}\ln[(x_1 - x_2)^2 + (y_1 - y_2)^2] + C\frac{\Gamma_2^i}{x_1 - x_2}\tan^{-1}\frac{y_1 - y_2}{x_1 - x_2} + f(x_1) + g(x_2, y_2),$$

But

$$\begin{aligned} -C\frac{\partial H}{\partial x_1} &= -\frac{1}{2\pi}\frac{\Gamma_2^r(x_1 - x_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2} + \frac{\Gamma_2^i}{(x_1 - x_2)^2}\tan^{-1}\frac{y_1 - y_2}{x_1 - x_2} + \\ &+ \frac{\Gamma_2^i}{x_1 - x_2}\frac{y_1 - y_2}{(x_1 - x_2)^2}\frac{(x_1 - x_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2} + f'(x_1) \neq \\ &\neq \frac{1}{2\pi}\frac{\Gamma_2^r(x_1 - x_2) - \Gamma_2^r(y_1 - y_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \end{aligned}$$

This argument can be generalized for non canonical symplectic structures.

But even though the system is not Hamiltonian, it still has some symmetries.Since the system is translation, rotation and scale invariant, it suggest that some kind of linear and angular momenta should be preserved. We now show that close analog of ”moment of vorticity” is indeed preserved for the point singularities. Clearly

$$\sum_{\alpha=1}^{N} \Gamma_\alpha \dot{z}_\alpha^* = \frac{1}{2\pi i} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N}{}' \frac{\Gamma_\alpha \Gamma_\beta}{z_\alpha - z_\beta} = 0.$$

Thus

$$I = \sum_{\alpha=1}^{N} \Gamma_\alpha^* z_\alpha,$$

is a conserved quantity. We call it *conjugate moment of intensity*.

Additionally, since the equations of motion can be rewritten as

$$\frac{d}{dt}(z_\alpha - z_\gamma)^* = \sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta} - \sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\gamma - z_\beta},$$

or

$$\dot{l}_{\alpha\gamma}^* = \sum_{\beta=1}^{N}{}' \Gamma_\beta \left(\frac{1}{l_{\alpha\beta}} - \frac{1}{l_{\gamma\beta}} \right), \tag{4.2.1}$$

where $l_{\alpha\beta} = z_\alpha - z_\beta$ is a vector connecting z_α and z_β. This allows reduction of the system by one complex variable (or by two real). The reconstruction of the original variables can be done as follows

$$\begin{aligned} z_1 &= 0, \\ z_2 &= z_1 - l_{12}, \\ &\dots, \\ z_N &= z_{N-1} - l_{N-1,N}. \end{aligned} \tag{4.2.2}$$

This means that we can fix one of the singularities at the origin. The $l_{\alpha\beta}$ variable no longer have the translational symmetry. But the rotational symmetry is still present and gives the preservation of conjugate moment of intensity.

Chapter 5

Fixed equilibria

In order to study fixed equilibria we use ideas developed in the first part. The configuration matrix approach is used to prove existence and uniqueness results. The singular value decomposition and Shannon entropy are used to find and characterize fixed equilibria. More detailed study is performed for $N = 2, 3, 4$ point singularities fixed equilibria, collinear fixed equilibria and equilibria along prescribed curves.

5.1 Existence and uniqueness

As we have shown earlier, equations of point singularity evolution in 2D are

$$\dot{z}_\alpha^* = \frac{1}{2\pi i} \sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta}; \quad z_\alpha(t) \equiv x_\alpha(t) + i y_\alpha(t) \in \mathbb{C}, \quad \Gamma_\beta \in \mathbb{C}, \tag{5.1.1}$$

In order to find fixed equilibria we have to find solutions of point singularity equations (5.1.1) for which $\dot{z}_\alpha^*(t) = 0$. For this, we have the N coupled equations:

$$\sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta} = 0, \quad (\alpha = 1, ...N), \tag{5.1.2}$$

where we are interested in positions z_α and strengths Γ_α for which this nonlinear algebraic system is satisfied. Since Eqn (5.1.2) is *linear* in the Γ's, it can more productively be written in matrix form

$$A\mathbf{\Gamma} = 0 \tag{5.1.3}$$

where $A \in \mathbb{C}^{N\times N}$ is evidently a skew-symmetric matrix $A = -A^T$, with entries $[a_{\alpha\alpha}] = 0$, $[a_{\alpha\beta}] = \frac{1}{z_\alpha - z_\beta} = -[a_{\beta\alpha}]$. We call A the *configuration matrix* associated with the interacting particle system (5.1.1). The collection of points $\{z_1(0), z_2(0), ..., z_N(0)\}$ in the complex plane is called the *configuration*.

From (5.1.3), we can conclude that the points z_α are in a fixed equilibrium configuration if $\det(A) = 0$, i.e. there is at least one zero eigenvalue of A. If the corresponding eigenvector is real, the configuration is made up of point-vortices. If it is imaginary, it is made up of sources and sinks. If it is complex, it is made up of spiral sources and sinks. Notice also that if $\frac{dz_\alpha^*}{dt} = 0$, then one can prove that $\frac{d^n z_\alpha^*}{dt^n} = 0$ for any n. It follows that:

Theorem 5.1.1. *For a given configuration of N points $\{z_1, z_2, ..., z_N\}$ in the complex plane, there exists a set of singularity strengths $\mathbf{\Gamma}$ for which the configuration is a fixed equilibrium solution of the dynamical system (4.1.7) iff A has a kernel, or equivalently, if there is at least one zero eigenvalue of A. If the nullspace dimension of A is one, i.e. there is only one zero eigenvalue, the choice of $\mathbf{\Gamma}$ is unique (up to a multiplicative constant). If the nullspace dimension is greater than one, the choice of $\mathbf{\Gamma}$ is not unique and can be any linear combination of the basis elements of null(A).*

Since A is skew-symmetric, it follows that

$$\det(A) = \det(-A^T) = (-1)^N \det(A^T) = \det(A^T). \tag{5.1.4}$$

Hence, for N *odd*, we have $-\det(A^T) = \det(A^T)$, which implies $\det(A^T) = 0$.

Theorem 5.1.2. *When N is odd, A always has at least one zero eigenvalue, hence for any configuration there exists a choice $\mathbf{\Gamma} \in \mathbb{C}$ for which the system is a fixed equilibrium.*

When N is even, there may or may not be a fixed equilibrium, depending on whether or not A has a non-trivial nullspace. In general, we would like to determine a basis set for the nullspace of A for a given configuration, i.e. the set of all strengths for which a given configuration remains fixed. Other important general properties of skew-symmetric matrices are listed below:

1. The eigenvalues always come in pairs $\pm\lambda$. If N is odd, there is one unpaired eigenvalue that is zero.

2. If N is even, $\det(A) = Pf(A)^2 \geq 0$, where Pf is the Pfaffian.

3. Real skew-symmetric matrices have pure imaginary eigenvalues.

Recall that every matrix can be written as the sum of a Hermitian matrix ($B = B^\dagger$) and a skew-Hermitian matrix ($C = -C^\dagger$). To see this, notice

$$A \equiv \frac{1}{2}(A + A^\dagger) + \frac{1}{2}(A - A^\dagger). \tag{5.1.5}$$

Here, $B \equiv \frac{1}{2}(A + A^\dagger) = B^\dagger$ and $C \equiv \frac{1}{2}(A - A^\dagger) = -C^\dagger$. A matrix is *normal* if $AA^\dagger = A^\dagger A$, otherwise it is *non-normal*. If we calculate $AA^\dagger - A^\dagger A$, where $A = B + C$ as above, then it is easy to see that

$$AA^\dagger - A^\dagger A = 2(CB - BC). \tag{5.1.6}$$

Therefore, if $B = 0$ or $C = 0$, A is normal.

Theorem 5.1.3. *All Hermitian or skew-Hermitian matrices are normal.*

The generic configuration matrix A arising from (5.1.3) is, however, non-normal.

For normal matrices, the following spectral-decomposition holds:

Theorem 5.1.4. $\mathbf{A}$ *is a normal matrix* $\Leftrightarrow$ $\mathbf{A}$ *is unitarily diagonalizable, i.e.*

$$\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^\dagger \tag{5.1.7}$$

where $\mathbf{Q}$ *is unitary.*

Here, the columns of $\mathbf{Q}$ are the N linearly independent eigenvectors of A that can be made mutually orthogonal. The matrix Λ is a diagonal matrix with the N eigenvalues down the diagonal[1].

In general, however, for the system of interacting particles governed by (5.1.2), (5.1.3), $A \in \mathbb{C}^{N\times N}$ will be a non-normal matrix. The most comprehensive decomposition of A in this case is the singular value decomposition described in section 2.2. Because of (5.1.3), we seek configuration matrices with one or more singular values that are zero.

5.2 Collinear equilibria

For the special case in which all the particles lie on a straight line, there is no loss in assuming $z_\alpha = x_\alpha \in \mathbb{R}$. Then $A \in \mathbb{R}^{N\times N}$, A is a normal skew-symmetric matrix, and the eigenvalues are pure imaginary. As an example, consider the collinear case $N = 3$. Let the particle positions be $x_1 < x_2 < x_3$, with corresponding strengths Γ_1, Γ_2, Γ_3. The A matrix is then given by

$$A = \begin{bmatrix} 0 & \frac{1}{x_1-x_2} & \frac{1}{x_1-x_3} \\ \frac{1}{x_2-x_1} & 0 & \frac{1}{x_2-x_3} \\ \frac{1}{x_3-x_1} & \frac{1}{x_3-x_2} & 0 \end{bmatrix}. \tag{5.2.1}$$

Since N is odd, we have $\det(A) = 0$. The other two eigenvalues are given by:

$$\lambda_{123} = \pm i\sqrt{\frac{1}{(x_2-x_1)^2} + \frac{1}{(x_3-x_2)^2} + \frac{1}{(x_3-x_1)^2}}, \tag{5.2.2}$$

[1]See [GVL96] for details.

which is invariant under cyclic permutations of the indices ($\lambda_{123} = \lambda_{231} = \lambda_{312}$). We can scale the length of the configuration so that the distance between x_1 and x_3 is one, hence without loss of generality, let $x_1 = 0, x_2 = x, x_3 = 1$. The other two eigenvalues are then given by the formula:

$$\lambda = \pm i\sqrt{\frac{(1-x+x^2)^2}{x^2(1-x)^2}}. \tag{5.2.3}$$

It is easy to see that the numerator has no roots in the interval $(0, 1)$, hence the nullspace dimension of A is one. The nullspace vector is then given (uniquely up to multiplicative constant) by:

$$\mathbf{\Gamma} = \begin{bmatrix} 1 \\ -\left(\frac{x_3-x_2}{x_3-x_1}\right) \\ \left(\frac{x_3-x_2}{x_2-x_1}\right) \end{bmatrix}. \tag{5.2.4}$$

For the special symmetric case $x_3 - x_1 = 1$, $x_3 - x_2 = 1/2$, $x_2 - x_1 = 1/2$, we have $\Gamma_1 = 1, \Gamma_2 = -1/2, \Gamma_3 = 1$. We show this case in figure 5.1 along with the separatrices associated with the corresponding flowfield generated by the singularities. Since the sum of the strengths of the three vortices is $\Gamma_1 + \Gamma_2 + \Gamma_3 = 1 - 1/2 + 1 = 3/2$, the far field is that of a point vortex of strength $\Gamma = 3/2$. Interestingly, for the collinear cases, since A is real, the nullspace vector is either real, or if multiplied by i, is pure imaginary. Hence, each collinear configuration of point vortices obtained with a given $\mathbf{\Gamma} \in \mathbb{R}$ is also a collinear configuration of sources/sinks with corresponding strengths given by $i\mathbf{\Gamma}$. The corresponding streamline pattern for the source/sink configuration, as shown in the dashed curves of figure 5.1, is the orthogonal complement of the curves corresponding to the point vortex case.

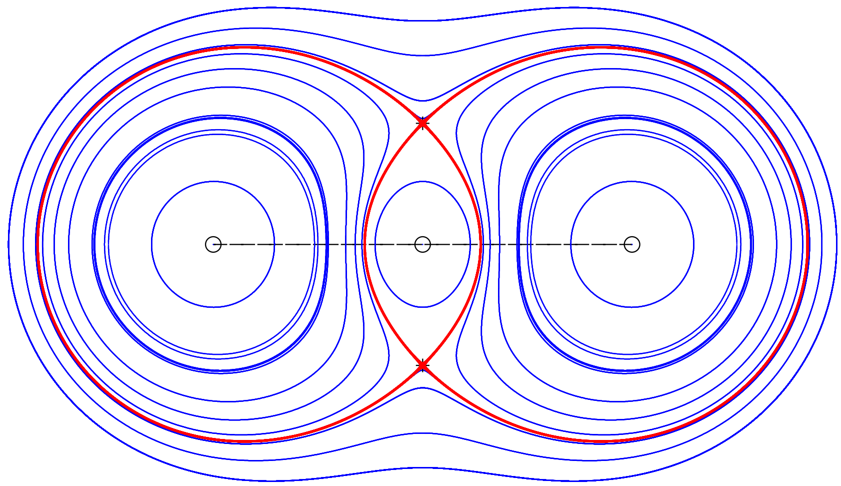


Figure 5.1: $N = 3$ evenly distributed point vortices on a line with strengths $\Gamma_1 = 1, \Gamma_2 = -\frac{1}{2}, \Gamma_3 = 1$, in equilibrium. The far field is that of a point vortex at the center-of-vorticity of the system. Solid streamline pattern is for point vortices, dashed streamline pattern is for source/sink system. The patterns are orthogonal.

For N even, we cannot say a priori whether or not $\det(A) = 0$ as the case for $N = 2$ shows. For this, the A matrix is

$$A = \begin{bmatrix} 0 & \frac{1}{x_1 - x_2} \\ \frac{1}{x_2 - x_1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{d} \\ -\frac{1}{d} & 0 \end{bmatrix}. \tag{5.2.5}$$

The eigenvalues are $\lambda = \pm i/d$, hence there is no equilibrium (except in the limit $d \to \infty$).

We show in figures 5.2 and 5.3 two representative examples of collinear fixed point vortex equilibria for $N = 7$, along with their corresponding global streamline patterns. In figure 5.2 we deposit seven evenly spaced points on a line and solve for the nullspace vector to obtain the singularity strengths (ordered from left to right)

$$\mathbf{\Gamma} = (1.0000, -0.5536, 0.9212, -0.5797, 0.9212, -0.5536, 1.0000), \tag{5.2.6}$$

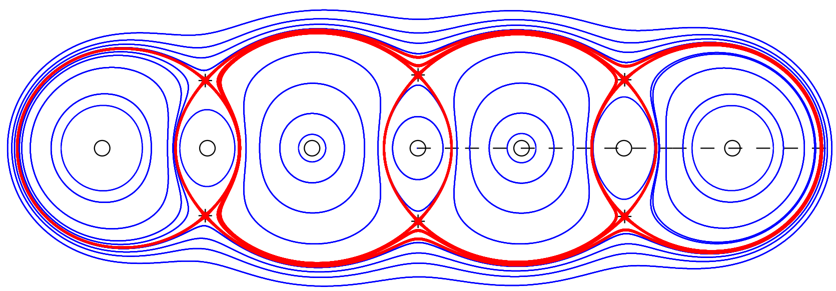


Figure 5.2: $N = 7$ evenly distributed point vortices on a line. The far field is that of a point vortex at the center-of-vorticity of the system. Because of the symmetry of the spacing, the vortex strengths are symmetric about the central point x_4 which also corresponds to the center-of-vorticity.

$$\sum_\alpha \Gamma_\alpha = 2.1555. \tag{5.2.7}$$

Because of the even spacing, the strengths are symmetric about the central point x_4 ($\Gamma_1 = \Gamma_7, \Gamma_2 = \Gamma_6, \Gamma_3 = \Gamma_5$), which is also the location of the center-of-vorticity $\sum_{\alpha=1}^{7} \Gamma_\alpha x_\alpha$. Figure 5.3 shows a fixed equilibrium corresponding to seven points randomly placed on a line. The nullspace vector for this case is (ordered from left to right)

$$\mathbf{\Gamma} = (1.0000, -0.5071, 0.5342, -0.4007, 0.2815, -0.2505, 1.0743), \tag{5.2.8}$$

$$\sum_\alpha \Gamma_\alpha = 1.7317. \tag{5.2.9}$$

In both cases, the singularities are all point vortices (or source/sink systems) hence are examples of collinear equilibria such as those discussed in [Are07a, Are07b, Are09] and [ANS$^+$02] where the strengths are typically chosen as equal. The streamline pattern at

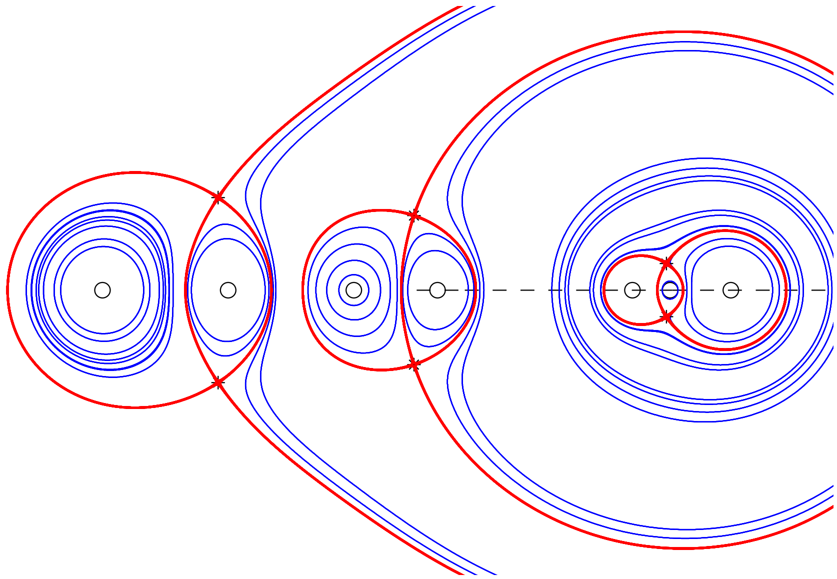


Figure 5.3: $N = 7$ randomly distributed point vortices on a line. The far field is that of a point vortex at the center-of-vorticity of the system.

infinity in both cases is that of a single point vortex of strength $\sum_{\alpha=1}^{7} \Gamma_\alpha \neq 0$ located at the center of vorticity $\sum_{\alpha=1}^{7} \Gamma_\alpha x_\alpha$.

5.3 Triangular equilibria

The case $N = 3$ is somewhat special and worth treating separately. Given any three points $\{z_1, z_2, z_3\}$ in the complex plane, the corresponding configuration matrix A is:

$$A = \begin{bmatrix} 0 & \frac{1}{z_1 - z_2} & \frac{1}{z_1 - z_3} \\ \frac{1}{z_2 - z_1} & 0 & \frac{1}{z_2 - z_3} \\ \frac{1}{z_3 - z_1} & \frac{1}{z_3 - z_2} & 0 \end{bmatrix}. \tag{5.3.1}$$

There is no loss of generality in choosing two of the points along the real axis, one at the origin of our coordinate system, the other at $x = 1$. Hence we set $z_1 = 0$, $z_2 = 1$, and we let $z_3 \equiv z$. Then A is written much more simply:

$$A = \begin{bmatrix} 0 & -1 & -\frac{1}{z} \\ 1 & 0 & \frac{1}{1-z} \\ \frac{1}{z} & \frac{1}{z-1} & 0 \end{bmatrix}. \tag{5.3.2}$$

Since N is odd, one of the eigenvalues of A is zero. The other two are given by:

$$\lambda = \pm i\sqrt{\frac{1}{z^2} + \frac{1}{(1-z)^2} + 1} = \pm i\sqrt{\frac{(1-z+z^2)^2}{z^2(1-z)^2}} \tag{5.3.3}$$

When the numerator is not zero, the nullspace dimension is one and it is easy to see that the nullspace of A is given by:

$$\mathbf{\Gamma} = \begin{bmatrix} \frac{1}{z-1} \\ -\frac{1}{z} \\ 1 \end{bmatrix}. \tag{5.3.4}$$

However, the numerator is zero at the points:

$$z = \exp(\frac{\pi i}{3}), \exp(\frac{5\pi i}{3}), \tag{5.3.5}$$

at which $\operatorname{Re} z = \frac{1}{2}$, $\operatorname{Im} z = \pm\frac{\sqrt{3}}{2}$. This forms an equilateral triangle in which case the nullspace dimension is three. We have thus proven the following:

Theorem 5. *For three point vortices, or for three sources/sinks, the only fixed equilibria are collinear. For any three point singularities the nullspace dimension of A is one and is given by (5.3.7).*

We show a fixed equilibrium equilateral triangle state in figure 5.4 along with the corresponding streamline pattern.

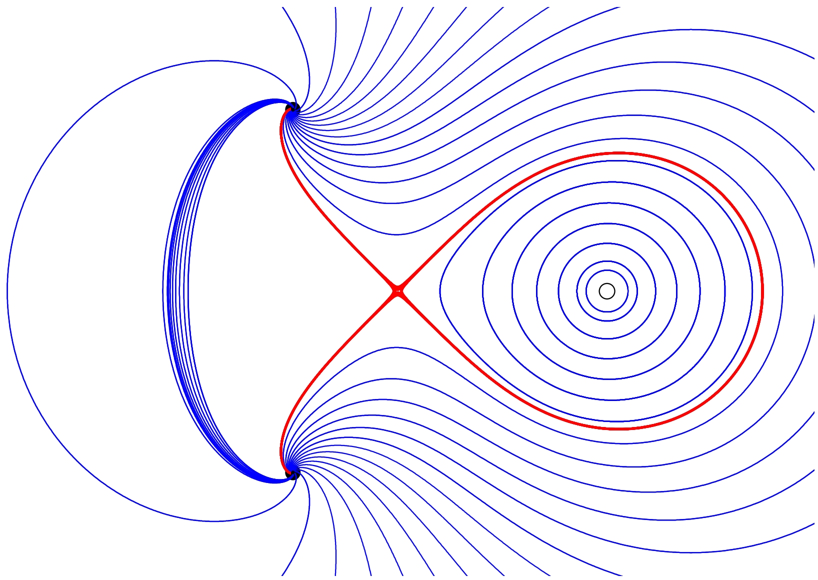


Figure 5.4: $N = 3$ equilateral triangle configuration with corresponding streamline pattern. The strengths are given by $\Gamma_1 = 1.0000$, $\Gamma_2 = -0.5000 + 0.8660i$, $\Gamma_3 = -0.5000 + 0.8660i$.

Equilateral triangular equilibria

As it was pointed before, equilateral triangle has all of the eigenvalues equal to zero. Without loss of generality assume positions of point singularities are located at $0, 1, e^{\frac{\pi i}{3}}$. Then

$$A = \begin{bmatrix} 0 & -1 & -e^{-\frac{\pi i}{3}} \\ e^{-\frac{\pi i}{3}} & 0 & \frac{1}{1-e^{\frac{\pi i}{3}}} \\ 1 & -\frac{1}{1-e^{\frac{\pi i}{3}}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -\frac{1}{2}+\frac{\sqrt{3}}{2} \\ -1 & 0 & -\frac{1}{2}-\frac{\sqrt{3}}{2} \\ \frac{1}{2}-\frac{\sqrt{3}}{2} & \frac{1}{2}+\frac{\sqrt{3}}{2} & 0 \end{bmatrix}. \qquad (5.3.6)$$

The three eigenvalues are 0. The nullspace of A is covered by

$$\mathbf{\Gamma} = \begin{bmatrix} \frac{1}{z-1} \\ -\frac{1}{z} \\ 1 \end{bmatrix}. \tag{5.3.7}$$

And since the characteristic equation for the matrix A is $\lambda^3 = 0$, the configuration matrix is nilpotent. Thus the two vectors from the orthogonal complement to the null space are in lying in the invariant subspace of matrix A.

5.4 Equilibria along prescribed curves

We now ask a more general and interesting question. Given any curve in the complex plane, if we distribute points $\{z_\alpha\}$, $(\alpha = 1, ..., N)$ along the curve, is it possible to find a strength vector $\mathbf{\Gamma}$ so that the configuration is fixed? The answer is yes, if N is odd, and sometimes, if N is even.

Figures 5.5 - 5.11 show a collection of fixed equilibria along curves that we prescribe. First, figure 5.5 shows 7 points places randomly in the plane, with the singularity strengths obtained from the nullspace of A so that the system is an equilibrium. The strengths are given by: $\mathbf{\Gamma} = (1.0000, -0.7958 + 1.0089i, -1.3563 - 0.4012i, 0.0297 + 0.1594i, 0.9155 + 0.3458i, -2.0504 - 0.8776i, -0.1935 - 1.0802i)^T$ with the sum given by $-2.4508 - 0.8449i$. Thus, the far field is that of a spiral-sink configuration. Figure 5.6 shows the case of $N = 7$ points distributed evenly around a circle. The nullspace vector is given by $\mathbf{\Gamma} = (1.0000, -0.9010 + 0.4339i, 0.6235 - 0.7818i, -0.2225 + 0.9749i, -0.2225 - 0.9749i, 0.6235 + 0.7818i, -0.9010 - 0.4339i)^T$. For this very symmetric case, the sum of the strengths is zero, hence in a sense, the far field vanishes. Figure 5.7 shows the case of $N = 7$ points placed at random positions on a circle. Here, the nullspace vector is given by $\mathbf{\Gamma} = (1.0000, -0.6342 + 0.4086i, 0.3699 - 0.5929i, -0.1501+0.6135i, -0.2483-0.9884i, 0.2901+0.3056i, -0.3595-0.2686i)^T$ The random placement of points breaks the symmetry of the previous case and the sum of strengths is given by $0.2649 - 0.5222i$ which corresponds to a spiral-sink. In figures 5.8 and 5.9 we show equilibrium distribution of points along a curve we call a 'flower-petal', given by the formula $r(\theta) = \cos(2\theta)$, $0 \leq \theta \leq 2\pi$. In figure 5.8 we distribute them evenly on the curve, while in figure 5.9 we distribute them randomly. The particle strengths from the configuration in figure 5.8 are $\mathbf{\Gamma} = (1.0000, 0.1824 + 0.1498i, -0.9892 - 0.9103i, -0.1378 - 0.5333i, -0.1378 + 0.5333i, -0.9892 + 0.9103i, 0.1824 - 0.1498i)^T$ with sum equaling -0.8892 corresponding to a far field point vortex. Figure 5.9 shows particles distributed randomly on the same flower-petal curve. Here, the particle strengths are $\mathbf{\Gamma} = (1.0000, 0.2094 - 0.4071i, -0.3009 + 0.3003i, 0.0404 - 0.2864i, -0.1779 + 0.2773i, 0.4236 + 0.8052i, -0.4702 - 0.3304i)^T$, with sum given by $.7244 + .3589i$. Hence the far field corresponds to a source-spiral.

The last two configurations, shown in figures 5.10 and 5.11 are equilibria distributed along figure eight curves, given by the formulas $r(\theta) = \cos^2(\theta)$, $0 \leq \theta \leq 2\pi$. In figure 5.10 we distribute the points evenly around the curve, which gives rise to strengths $\mathbf{\Gamma} = (1.0000, -0.2734 + 0.5350i, 0.0239 - 0.2080i, 0.1063 - 0.0517i, 0.1063 + 0.0517i, 0.0239 + 0.2080i, -0.2734 - 0.5350i)^T$, whose sum is $.7136$, thus a far field point vortex. In contrast, when the points are distributed randomly around the same curve, as in figure 5.11, the strengths are given by $\mathbf{\Gamma} = (1.0000, -0.1054 + 0.5724i, -0.0174 - 0.4587i, 0.9208 + 1.2450i, -0.0460 -$

$0.4577i, -0.5292 + 0.2371i, -0.2543 - 0.0921i)^T$, with sum equaling $.9685 + 1.0460i$, hence a far field source-spiral.

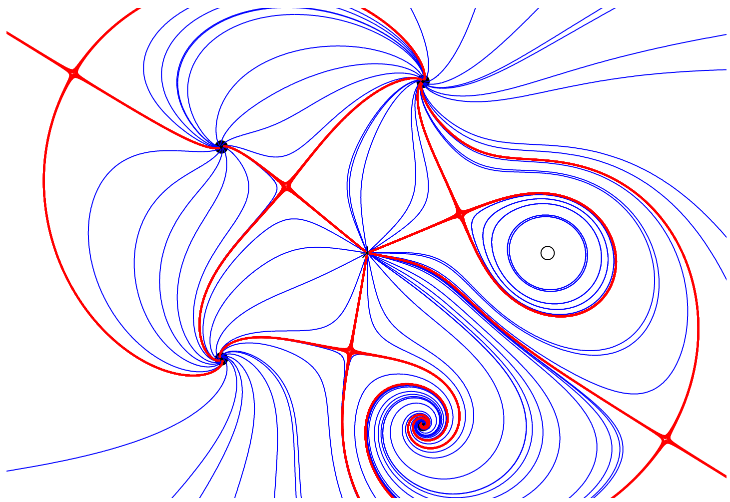


Figure 5.5: Fixed equilibrium for seven points placed at random locations in the plane. The far field is a spiral-sink (figure 1(e)) with since $\sum \Gamma_\alpha = -2.4508 - 0.8449i$.

5.5 Even number of singularities

As we showed before, point singularity configuration will be a fixed equilibria if $\det A = 0$. For even dimensional ($N = 2n$) skew-symmetric matrix $A = (a_{i,j})_{1 \leq i,j \leq 2n}$ determinant is equal to

$$\det A = (\mathrm{pf}(A))^2, \tag{5.5.1}$$

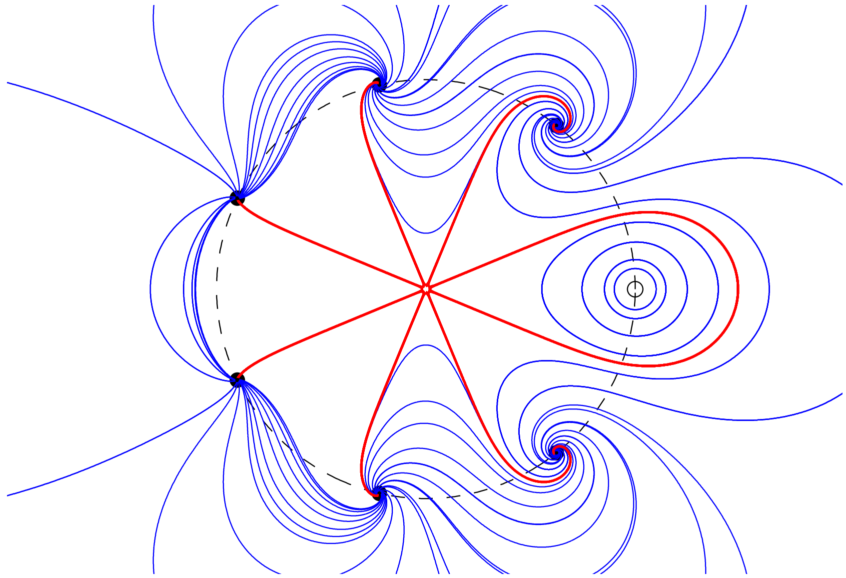


Figure 5.6: $N = 7$ evenly distributed points on a circle (dashed curve) in equilibrium. Because of the symmetry of the configuration, $\sum \Gamma_\alpha = 0$, hence the far-field vanishes.

where $\mathrm{pf}(A)$ is a Pfaffian of matrix A defined as

$$\mathrm{pf}(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sign}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)} =, \tag{5.5.2}$$

with S_{2n} being symmetric group and $\mathrm{sign}(\sigma)$ signature of σ. Alternative definition which we will use is

$$\mathrm{pf}(A) = \sum_{i=2}^{2n} (-1)^i a_{1,i} \mathrm{pf}(A_{\hat{1},\hat{i}}), \tag{5.5.3}$$

where $A_{\hat{1},\hat{i}}$ is matrix A with 1st and ith row and column removed. Also, by convention Pfaffian of 0×0 matrix is 1.

From (5.5.1) we see that first order root of Pfaffian is second order root of determinant. Thus even dimensional matrix with Pfaffian equal to 0 has two dimensional null space.

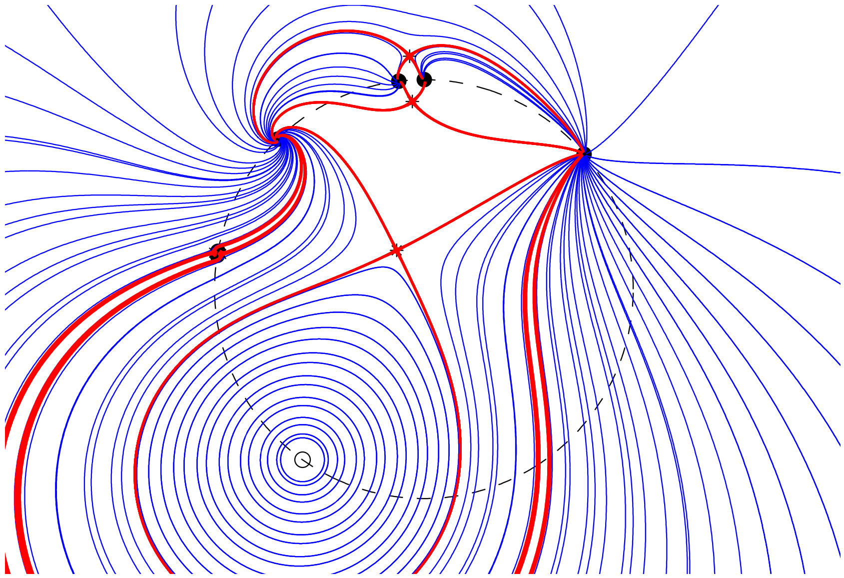


Figure 5.7: $N = 7$ randomly distributed particles on a circle (dashed curve) in equilibrium along with the corresponding streamline pattern. The far field streamline pattern is that of a spiral-sink (figure 1(g)) since $\sum \Gamma_\alpha = 0.2649 - 0.5222i$.

Four singularities

Since every even-dimensional skew symmetric matrix has paired eigenvalues (if λ is its eigenvalue, then $-\lambda$ is an eigenvalue as well) we can have either two dimensional or four dimensional null space. Four dimensional null space has only zero matrix which is not a configuration matrix for the singularity equilibrium. Thus we can have only two dimensional null space.

As we have said before, determinant of even dimensional skew symmetric matrix is a square of the Pfaffian of the matrix. This will give us following condition for the positions of singularities

$$\frac{1}{(z_2 - z_3)^2 (z_1 - z_4)^2} + \frac{1}{(z_1 - z_3)^2 (z_2 - z_4)^2} + \frac{1}{(z_1 - z_2)^2 (z_3 - z_4)^2} = 0. \quad (5.5.4)$$

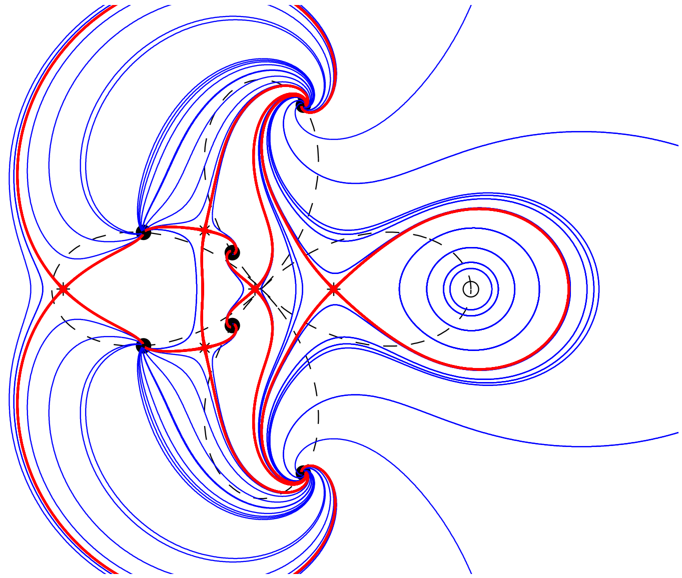


Figure 5.8: $N = 7$ evenly distributed particles in equilibrium on the curve $r(\theta) = \cos(2\theta)$ (dashed curve) along with the corresponding streamline pattern. The far field corresponds to a point vortex since $\sum \Gamma_\alpha = -0.8892$.

Since the equilibrium configuration of singularities is translation, rotation and scale invariant, without loss of generality we can assume $z_1 = 0, z_2 = 1$. Then if we choose $z_3 = a$ we will get following equation for the fourth position $z_4 = x$

$$\frac{1}{(a-y)^2} + \frac{1}{a^2(y-1)^2} + \frac{1}{(a-1)^2 y^2} = 0. \tag{5.5.5}$$

The solutions are

$$z_4 = \frac{a + a^2 \pm \sqrt{3}\sqrt{a^2 - 2a^3 - a^4}}{2\left(1 - a + a^2\right)}. \tag{5.5.6}$$

Thus we proved

Theorem 5.5.1. *For any initial positions of three out four point singularities there exist at least one position for the fourth one. For any configuration of four point singularities the null space dimension is two.*

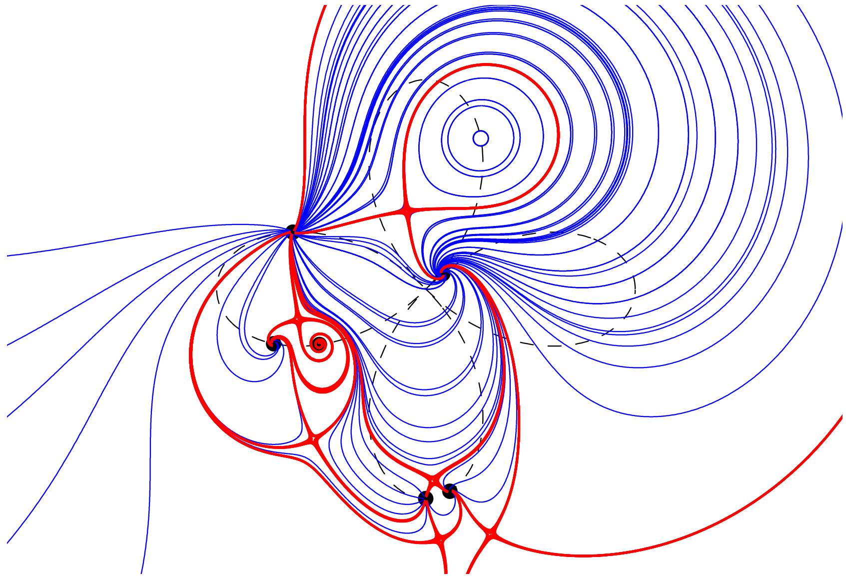


Figure 5.9: $N = 7$ randomly distributed particles in equilibrium on the curve $r(\theta) = \cos(2\theta)$ (dashed curve). The far field corresponds to a source-spiral (figure 1(f)) since $\sum \Gamma_\alpha = 0.7244 + 0.3589i$.

For example, if we chose $z_3 = 1+i$, then $z_4 = \frac{3}{2} - \frac{\sqrt{3}}{2} + i\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$. Demonstration of this configuration is given on Figure 5.12.

Six and more singularities

Using the same technique as above, we can build the equation for one unknown coordinate. First $N-1$ we can choose at random and then since the equation for the unknown coordinate can be solved in complex numbers, the fundamental theorem of algebra guaranties that at leas one solution exist. Thus for even N there are many fixed equilibria of point singularities.

We give some examples of the configurations for $N = 6, 8$ on Figure 5.13 and Figure 5.14.

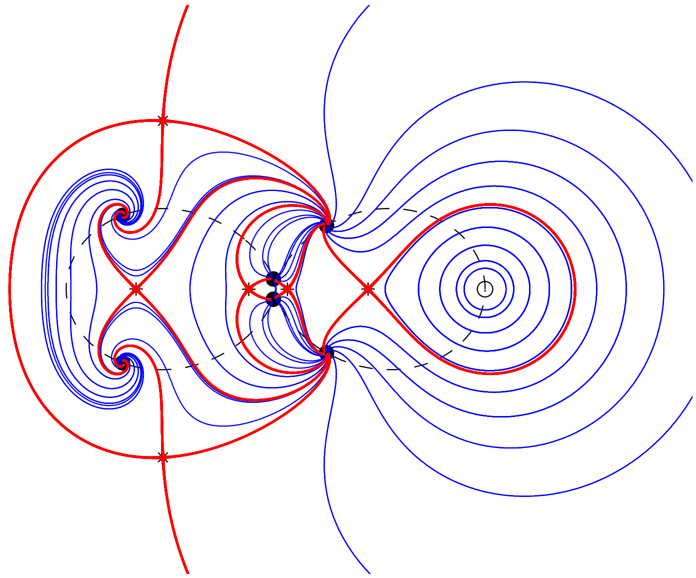


Figure 5.10: $N = 7$ evenly distributed particles in equilibrium on the curve $r(\theta) = \cos^2(\theta)$ (dashed curve). The far field corresponds to a point vortex since $\sum \Gamma_\alpha = 0.7136$.

5.6 Classification of equilibria in terms of the singular spectrum

Tables 5.1 - 5.6 show the complete singular spectrum for all the equilibria considered in this work. A common measure of 'robustness' associated with the configuration matrix, hence the equilibrium, is the size of the 'spectral gap' as measured by the size of the smallest non-zero singular value. From Table 5.2, the collinear state with points distributed randomly and the figure-eight state with points distributed evenly (Table 5.4) are the least robust in that their smallest non-zero singular values are closest to zero.

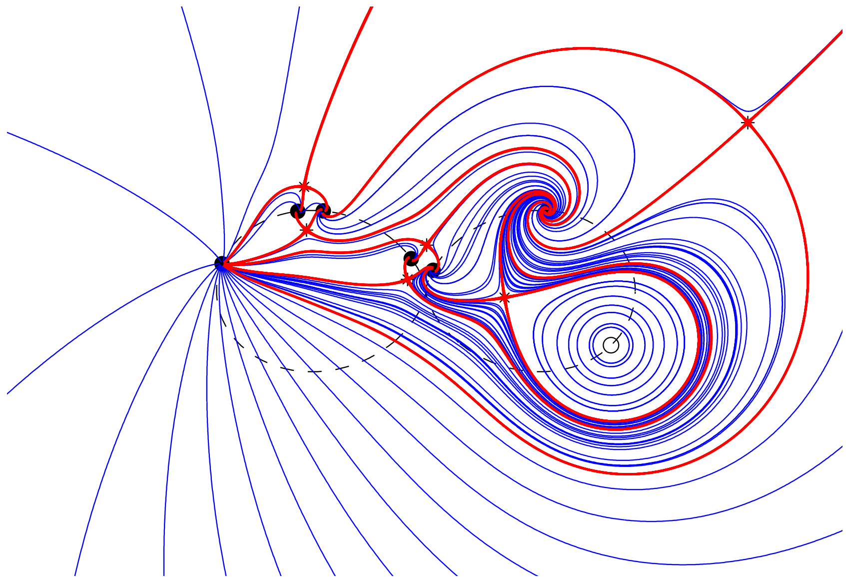


Figure 5.11: $N = 7$ randomly distributed particles in equilibrium on the curve $r(\theta) = \cos^2(\theta)$ (dashed curve). The far field corresponds to a source-spiral (figure 1(f)) since $\sum \Gamma_\alpha = 0.9685 + 1.0460i$.

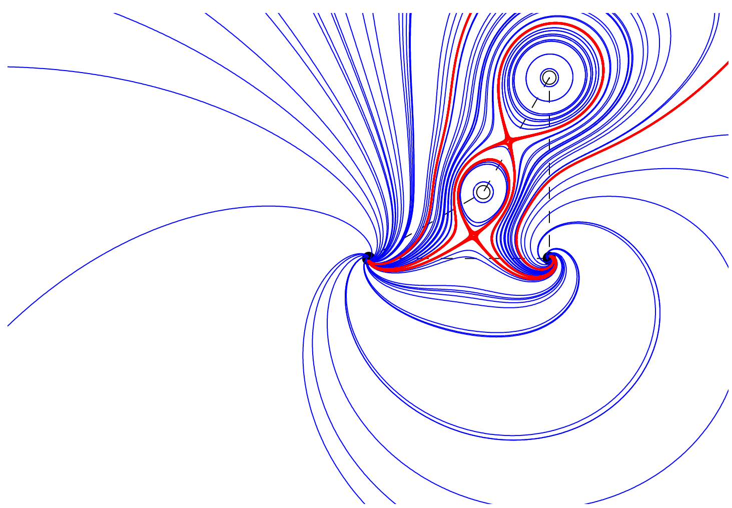


Figure 5.12: Fixed equilibrium for four points with one placed at random location in the plane. The far field is a spiral-sink (figure 1(e)) with since $\sum \Gamma_\alpha = -1.0490 - 1.1830i$.

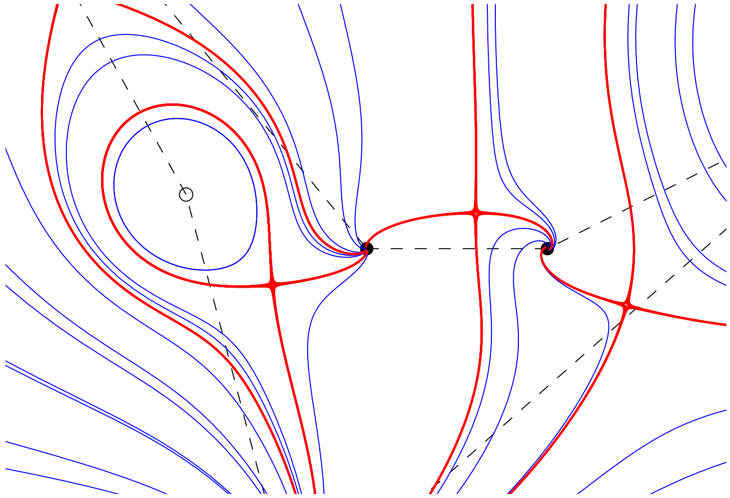


Figure 5.13: Fixed equilibrium for six points with five placed at random location in the plane. The far field is a spiral-sink (figure 1(e)) with since $\sum \Gamma_\alpha = -1.0881 - 1.3789i$.

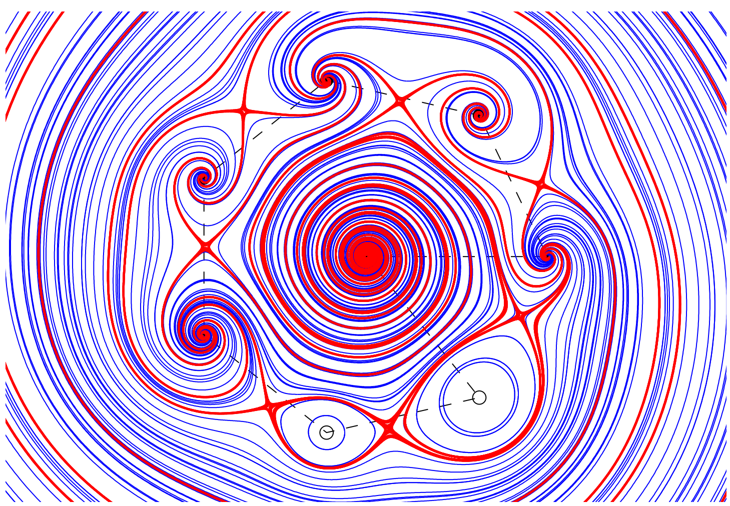


Figure 5.14: Fixed equilibrium for eight points with seven placed on a regular 7-gon. The far field is a spiral-sink (figure 1(e)) with since $\sum \Gamma_\alpha = 6.0000 - 0.4565i$.

Configuration	σ (unormalized)	σ (normalized)	Shannon entropy
	1.0000	0.5000	0.6931
Equilateral	1.0000	0.5000	
	0.00	0.00	
	1.0598	0.5000	0.6931
Isosceles (acute)	1.0598	0.5000	
	0.00	0.00	
	2.7203	0.5000	0.6931
Isosceles (obtuse)	2.7203	0.5000	
	0.00	0.00	
	1.2115	0.5000	0.6931
Arbitrary triangle	1.2115	0.5000	
	0.00	0.00	

Table 5.1: Singular spectrum of triangular states ($N = 3$)

Configuration	σ (unormalized)	σ (normalized)	Shannon entropy
	4.5000	0.5000	0.6931
$N = 3$	4.5000	0.5000	
	0.00	0.00	
	2.5249	0.3214	1.5237
$N = 7$ (even)	2.5249	0.3214	
	1.6831	0.1428	
	1.6831	0.1428	
	0.8420	0.0357	
	0.00	0.00	
	6.3408	0.4457	1.0723
$N = 7$ (random)	6.3408	0.4457	
	2.0969	0.0487	
	2.0969	0.0487	
	0.7062	0.0055	
	0.7062	0.0055	
	0.0000	0.0000	

Table 5.2: Singular spectrum of collinear states ($N = 3, 7$)

Configuration	σ (unormalized)	σ (normalized)	Shannon entropy
$N = 7$ (even)	3.0000	0.3214	1.5236
	3.0000	0.3214	
	2.0000	0.1429	
	2.0000	0.1429	
	1.0000	0.0357	
	1.0000	0.0357	
	0.0000	0.0000	
$N = 7$ (random)	3.7954	0.3363	1.4700
	3.7954	0.3363	
	2.4250	0.1373	
	2.4250	0.1373	
	1.0631	0.0264	
	1.0631	0.0264	
	0.0000	0.0000	

Table 5.3: Singular spectrum of circular states ($N = 7$)

Configuration	σ (unormalized)	σ (normalized)	Shannon entropy
$N = 7$ (even)	11.9630	0.4664	0.9651
	11.9630	0.4664	
	3.0001	0.0293	
	3.0001	0.0293	
	1.1454	0.0043	
	1.1454	0.0043	
	0.0000	0.0000	
$N = 7$ (random)	6.9337	0.3465	1.3929
	6.9337	0.3465	
	4.4357	0.1418	
	4.4357	0.1418	
	1.2769	0.0117	
	1.2769	0.0117	
	0.0000	0.0000	

Table 5.4: Singular spectrum of figure eight states ($N = 7$)

Configuration	σ (unormalized)	σ (normalized)	Shannon entropy
$N = 7$ (even)	5.9438	0.4447	1.1034
	5.9438	0.4447	
	1.8115	0.0413	
	1.8115	0.0413	
	1.0538	0.0140	
	1.0538	0.0140	
	0.0000	0.0000	
$N = 7$ (random)	8.0780	0.3875	1.3393
	8.0780	0.3875	
	3.8900	0.0899	
	3.8900	0.0899	
	1.9523	0.0226	
	1.9523	0.0226	
	0.0000	0.0000	

Table 5.5: Singular spectrum of flower states ($N = 7$)

Configuration	σ (unormalized)	σ (normalized)	Shannon entropy
$N = 4$ (random)	3.1566	0.5000	0.6931
	3.1566	0.5000	
	0.0000	0.0000	
	0.0000	0.0000	
$N = 6$ (random)	1.7723	0.4592	0.9758
	0.4592	0.3875	
	0.5283	0.0408	
	0.5283	0.0408	
	0.0000	0.0000	
	0.0000	0.0000	
$N = 8$ (symmetric)	4.0000	0.3810	1.3612
	4.0000	0.3810	
	2.0000	0.0952	
	2.0000	0.0952	
	1.0000	0.0238	
	1.0000	0.0238	
	0.0000	0.0000	
	0.0000	0.0000	

Table 5.6: Singular spectrum of even number states ($N = 4, 6, 8$)

Chapter 6

Stability

In this chapter we perform linear stability analysis of the fixed equilibrium configurations we found in the previous chapter.

6.1 Stability of triangular configurations

Without loss of generality we can assume $z_1^0 = (x_1^0, y_1^0) = (-1, 0)$, $z_2^0 = (x_2^0, y_2^0) = (1, 0)$ and $z_3^0 = (x_3^0, y_3^0) = (x, y)$ will be our changing parameters. Since all of the triangular configurations have one dimensional null space (except collinear, which we study in the next section) the parameter space is two dimensional.

From equations of motion (4.1.1) we have

$$\frac{1}{2\pi i}\sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta} =$$

$$= -\frac{i}{2\pi}\sum_{\beta=1}^{N}{}' \frac{(\Gamma_\beta^r + i\Gamma_\beta^i)((x_\alpha - x_\beta) + i(y_\alpha - y_\beta))}{(x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2} =$$

$$= \frac{1}{2\pi}\sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta^r(y_\alpha - y_\beta) + \Gamma_\beta^i(x_\alpha - x_\beta) - i(\Gamma_\beta^r(x_\alpha - x_\beta) - \Gamma_\beta^i(y_\alpha - y_\beta))}{(x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2}.$$

Thus

$$\dot{x}_\alpha = \frac{1}{2\pi}\sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta^i(x_\alpha - x_\beta) + \Gamma_\beta^r(y_\alpha - y_\beta)}{(x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2}, \tag{6.1.1}$$

$$\dot{y}_\alpha = \frac{1}{2\pi}\sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta^r(x_\alpha - x_\beta) - \Gamma_\beta^i(y_\alpha - y_\beta)}{(x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2}. \tag{6.1.2}$$

Matrix of the linearized system $M = (m_{ij})_{i,j=1,\dots,N}$ has the components

$$
\begin{aligned}
m_{\alpha\alpha} &= \sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta^i}{(x_\alpha^0 - x_\beta)^2 + (y_\alpha - y_\beta)^2} + \\
&+ \sum_{\beta=1}^{N}{}' \frac{2(x_\alpha^0 - x_\beta)(\Gamma_\beta^i(x_\alpha^0 - x_\beta) + \Gamma_\beta^r(y_\alpha - y_\beta))}{((x_\alpha^0 - x_\beta)^2 + (y_\alpha - y_\beta)^2)^2},\ \alpha = 1,\dots,N, \\
m_{\alpha\beta} &= \frac{\Gamma_\beta^i}{(x_\alpha^0 - x_\beta)^2 + (y_\alpha - y_\beta)^2} + \\
&+ \frac{2(x_\alpha^0 - x_\beta)(\Gamma_\beta^i(x_\alpha^0 - x_\beta) + \Gamma_\beta^r(y_\alpha - y_\beta))}{((x_\alpha^0 - x_\beta)^2 + (y_\alpha - y_\beta)^2)^2},\ \alpha, \beta = 1,\dots,N, \\
m_{\gamma\alpha} &= \sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta^r}{(x_\alpha^0 - x_\beta)^2 + (y_\alpha - y_\beta)^2} + \\
&+ \sum_{\beta=1}^{N}{}' \frac{2(x_\alpha^0 - x_\beta)(\Gamma_\beta^r(x_\alpha^0 - x_\beta) - \Gamma_\beta^i(y_\alpha - y_\beta))}{((x_\alpha^0 - x_\beta)^2 + (y_\alpha - y_\beta)^2)^2},\ \alpha = 1,\dots,N,\ \gamma = \alpha + N, \\
m_{\beta\alpha} &= \frac{\Gamma_\beta^r}{(x_\alpha^0 - x_\beta)^2 + (y_\alpha - y_\beta)^2} + \\
&+ \frac{2(x_\alpha^0 - x_\beta)(\Gamma_\beta^r(x_\alpha^0 - x_\beta) - \Gamma_\beta^i(y_\alpha - y_\beta))}{((x_\alpha^0 - x_\beta)^2 + (y_\alpha - y_\beta)^2)^2},\ \alpha = 1,\dots,N,\ \beta = N+1,\dots,2N, \\
m_{\alpha\gamma} &= \sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta^r}{(x_\alpha^0 - x_\beta)^2 + (y_\alpha - y_\beta)^2} + \\
&+ \sum_{\beta=1}^{N}{}' \frac{2(y_\alpha^0 - y_\beta)(\Gamma_\beta^r(x_\alpha - x_\beta) + \Gamma_\beta^r(y_\alpha^0 - y_\beta))}{((x_\alpha - x_\beta)^2 + (y_\alpha^0 - y_\beta)^2)^2},\ \alpha = 1,\dots,N,\ \gamma = \alpha + N, \\
m_{\alpha\beta} &= \frac{\Gamma_\beta^r}{(x_\alpha - x_\beta)^2 + (y_\alpha^0 - y_\beta)^2} + \\
&+ \frac{2(y_\alpha^0 - y_\beta)(\Gamma_\beta^i(x_\alpha - x_\beta) + \Gamma_\beta^r(y_\alpha^0 - y_\beta))}{((x_\alpha - x_\beta)^2 + (y_\alpha^0 - y_\beta)^2)^2},\ \alpha = 1,\dots,N,\ \beta = N+1,\dots,2N, \\
m_{\alpha\alpha} &= \sum_{\beta=1}^{N}{}' \frac{-\Gamma_\beta^i}{(x_\alpha - x_\beta)^2 + (y_\alpha^0 - y_\beta)^2} + \\
&+ \sum_{\beta=1}^{N}{}' \frac{2(y_\alpha^0 - y_\beta)(\Gamma_\beta^r(x_\alpha - x_\beta) - \Gamma_\beta^i(y_\alpha^0 - y_\beta))}{((x_\alpha - x_\beta)^2 + (y_\alpha^0 - y_\beta)^2)^2},\ \alpha = N+1,\dots,2N, \\
m_{\alpha\beta} &= \frac{-\Gamma_\beta^i}{(x_\alpha - x_\beta)^2 + (y_\alpha^0 - y_\beta)^2} + \qquad (6.1.3)
\end{aligned}
$$

$$+\frac{2(y_\alpha^0-y_\beta)(\Gamma_\beta^r(x_\alpha-x_\beta)-\Gamma_\beta^i(y_\alpha^0-y_\beta))}{((x_\alpha-x_\beta)^2+(y_\alpha^0-y_\beta)^2)^2},\ \alpha,\beta=N+1,\ldots,2N,$$

where $\Gamma_\alpha=\Gamma_\alpha^r+i\Gamma_\alpha^i$ is the intensity of α-th singularity.

By plugging in values for x_i^0 and y_i^0 and using the vector of intensities

$$\begin{aligned}\Gamma_1&=-\frac{-x_1+x_2-i(y_1+y_2)}{x_2-x_3+i(y_2-y_3)},\\ \Gamma_2&=-\frac{x_1-x_2+i(y_1-y_2)}{x_1-x_3+i(y_1-y_3)},\\ \Gamma_3&=1,\end{aligned}$$

we get the resulting matrix of linearized system

$$M_1=(M_{11}M_{12}),$$

where

$$M_{11}=\begin{pmatrix}\frac{y(5-6x+x^2+y^2)}{4\pi(1-2x+x^2+y^2)^2} & -\frac{y}{4\pi(1-2x+x^2+y^2)} & \frac{(-1+x)y}{\pi(1-2x+x^2+y^2)^2}\\ \frac{y}{4\pi(1+2x+x^2+y^2)} & -\frac{y(5+6x+x^2+y^2)}{4\pi(1+2x+x^2+y^2)^2} & \frac{(1+x)y}{\pi(1+2x+x^2+y^2)^2}\\ -\frac{y(-3+2x+x^2+y^2)}{\pi(1-2x+x^2+y^2)^2(1+2x+x^2+y^2)} & \frac{y(-3-2x+x^2+y^2)}{\pi(1-2x+x^2+y^2)(1+2x+x^2+y^2)^2} & \frac{8xy(-1+x^2+y^2)}{\pi(1-2x+x^2+y^2)^2(1+2x+x^2+y^2)^2}\\ \frac{-1+x+x^2-x^3+3y^2-xy^2}{4\pi(1-2x+x^2+y^2)^2} & \frac{-1+x}{4\pi(1-2x+x^2+y^2)} & \frac{1-2x+x^2-y^2}{2\pi(1-2x+x^2+y^2)^2}\\ -\frac{1+x}{4\pi(1+2x+x^2+y^2)} & \frac{-1+x^2+x^3+3y^2+x(-1+y^2)}{4\pi(1+2x+x^2+y^2)^2} & \frac{1+2x+x^2-y^2}{2\pi(1+2x+x^2+y^2)^2}\\ \frac{-1+x+x^2-x^3+3y^2-xy^2}{\pi(1-2x+x^2+y^2)^2(1+2x+x^2+y^2)} & \frac{-1+x^2+x^3+3y^2+x(-1+y^2)}{\pi(1-2x+x^2+y^2)(1+2x+x^2+y^2)^2} & \frac{2(1+x^4-2y^2-3y^4-2x^2(1+y^2))}{\pi(1-2x+x^2+y^2)^2(1+2x+x^2+y^2)^2}\end{pmatrix},$$

and

$$M_{12}=\begin{pmatrix}\frac{1-x^2+x^3-3y^2+x(-1+y^2)}{4\pi(1-2x+x^2+y^2)^2} & -\frac{-1+x}{4\pi(1-2x+x^2+y^2)} & \frac{-1+2x-x^2+y^2}{2\pi(1-2x+x^2+y^2)^2}\\ \frac{1+x}{4\pi(1+2x+x^2+y^2)} & -\frac{-1+x^2+x^3+3y^2+x(-1+y^2)}{4\pi(1+2x+x^2+y^2)^2} & -\frac{1+2x+x^2-y^2}{2\pi(1+2x+x^2+y^2)^2}\\ \frac{1-x^2+x^3-3y^2+x(-1+y^2)}{\pi(1-2x+x^2+y^2)^2(1+2x+x^2+y^2)} & -\frac{-1+x^2+x^3+3y^2+x(-1+y^2)}{\pi(1-2x+x^2+y^2)(1+2x+x^2+y^2)^2} & \frac{-2-2x^4+4y^2+6y^4+4x^2(1+y^2)}{\pi(1-2x+x^2+y^2)^2(1+2x+x^2+y^2)^2}\\ \frac{y(5-6x+x^2+y^2)}{4\pi(1-2x+x^2+y^2)^2} & -\frac{y}{4\pi(1-2x+x^2+y^2)} & \frac{(-1+x)y}{\pi(1-2x+x^2+y^2)^2}\\ \frac{y}{4\pi(1+2x+x^2+y^2)} & -\frac{y(5+6x+x^2+y^2)}{4\pi(1+2x+x^2+y^2)^2} & \frac{(1+x)y}{\pi(1+2x+x^2+y^2)^2}\\ -\frac{y(-3+2x+x^2+y^2)}{\pi(1-2x+x^2+y^2)^2(1+2x+x^2+y^2)} & \frac{y(-3-2x+x^2+y^2)}{\pi(1-2x+x^2+y^2)(1+2x+x^2+y^2)^2} & \frac{8xy(-1+x^2+y^2)}{\pi(1-2x+x^2+y^2)^2(1+2x+x^2+y^2)^2}\end{pmatrix}.$$

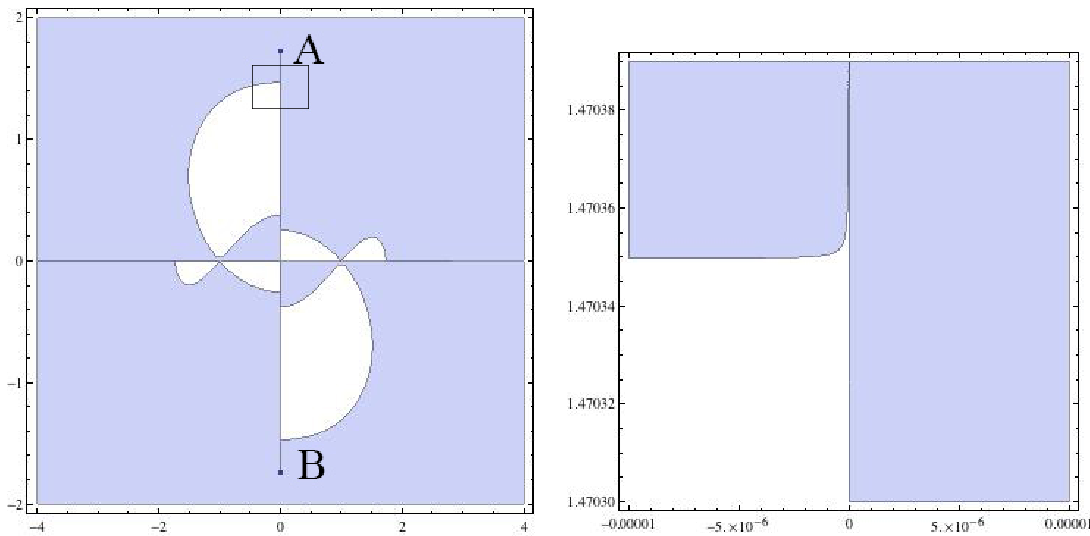


Figure 6.1: Unstable region for triangular configuration

The matrix is too complex for further analytic investigation. But we can find eigenvectors numerically for big enough region. The resulting region of linear instability is shown on Figure 6.1.

The points $A = (0, \sqrt{3})$ and $B = (0, \sqrt{3})$ represent *equilateral triangle* configurations. The eigenvalues for the equilateral triangle are

$$0, 0, 0, 0, 0, 0.$$

Thus the equilateral triangle configurations are neutrally stable.

6.2 Stability of collinear configurations

For the 3 singularities collinear equilibria, we can choose the first two to be $x_1 = 0$ and $x_2 = 1$. The last one will be changing parameter $x_3 = x$. The corresponding vector of intensities is

$$\Gamma_1 = \frac{x_1 - x_2}{x_2 - x_3}, \Gamma_2 = \frac{-x_1 + x_2}{x_1 - x_3}, \Gamma_3 = 0.$$

Then the matrix of linearized system is

$$M = \begin{pmatrix} 0 & 0 & 0 & -\frac{-1+x}{2\pi x^2} & \frac{1}{2\pi x} & -\frac{1}{2\pi x^2} \\ 0 & 0 & 0 & \frac{1}{2\pi - 2\pi x} & \frac{x}{2\pi(-1+x)^2} & -\frac{1}{2\pi(-1+x)^2} \\ 0 & 0 & 0 & \frac{1}{2\pi x^2 - 2\pi x^3} & \frac{1}{2\pi(-1+x)^2 x} & -\frac{1}{2\pi(-1+x)^2 x^2} \\ \frac{-1+x}{2\pi x^2} & -\frac{1}{2\pi x} & \frac{1}{2\pi x^2} & 0 & 0 & 0 \\ -\frac{1}{2\pi - 2\pi x} & -\frac{x}{2\pi(-1+x)^2} & \frac{1}{2\pi(-1+x)^2} & 0 & 0 & 0 \\ \frac{1}{2\pi(-1+x)x^2} & -\frac{1}{2\pi(-1+x)^2 x} & \frac{1}{2\pi(-1+x)^2 x^2} & 0 & 0 & 0 \end{pmatrix},$$

The eigenvalues are

$$0, 0, 0, 0, -\frac{3}{2\pi\sqrt{-x^2 + 2x^3 - x^4}}, \frac{3}{2\pi\sqrt{-x^2 + 2x^3 - x^4}}. \tag{6.2.1}$$

The roots of the denominator $-x^2 + 2x^3 - x^4$ are 0 and 1. And by checking the values in between we find that function is always negative. Thus we have

Theorem 6.2.1. *The collinear fixed equilibrium for 3 collinear point singularities is neutrally stable for any value of parameter x.*

Another interesting collinear equilibrium configuration is the case when $N = 5$ and point singularities are places symmetrically around the origin. Then let $x_1 = 0$, $x_2 = 1$,

$x_3 = -1$ and $x_4 = -x_5 = x$. After finding the vector of intensities and plugging it to the matrix of linearized system we get

$$M = \begin{pmatrix} 0 & M_{12} \\ M_{21} & 0 \end{pmatrix},$$

where

$$M_{12} = \begin{pmatrix} \frac{1-x^4}{\pi x^2+3\pi x^4} & \frac{3+x^2}{2\pi+6\pi x^2} & \frac{3+x^2}{2\pi+6\pi x^2} & -\frac{1}{2\pi x^2} & -\frac{1}{2\pi x^2} \\ \frac{1+14x^2+x^4}{4\pi+8\pi x^2-12\pi x^4} & \frac{3+11x^2+49x^4+x^6}{8\left(-1+x^2\right)^2\left(\pi+3\pi x^2\right)} & \frac{3+x^2}{8\pi+24\pi x^2} & -\frac{1}{2\pi(-1+x)^2} & -\frac{1}{2\pi(1+x)^2} \\ \frac{1+14x^2+x^4}{4\pi+8\pi x^2-12\pi x^4} & \frac{3+x^2}{8\pi+24\pi x^2} & \frac{3+11x^2+49x^4+x^6}{8\left(-1+x^2\right)^2\left(\pi+3\pi x^2\right)} & -\frac{1}{2\pi(1+x)^2} & -\frac{1}{2\pi(-1+x)^2} \\ \frac{1+14x^2+x^4}{4\pi x^2+8\pi x^4-12\pi x^6} & \frac{3+x^2}{2(-1+x)^2\left(\pi+3\pi x^2\right)} & \frac{3+x^2}{2(1+x)^2\left(\pi+3\pi x^2\right)} & -\frac{1+49x^2+11x^4+3x^6}{8x^2\left(-1+x^2\right)^2\left(\pi+3\pi x^2\right)} & -\frac{1}{8\pi x^2} \\ \frac{1+14x^2+x^4}{4\pi x^2+8\pi x^4-12\pi x^6} & \frac{3+x^2}{2(1+x)^2\left(\pi+3\pi x^2\right)} & \frac{3+x^2}{2(-1+x)^2\left(\pi+3\pi x^2\right)} & -\frac{1}{8\pi x^2} & -\frac{1+49x^2+11x^4+3x^6}{8x^2\left(-1+x^2\right)^2\left(\pi+3\pi x^2\right)} \end{pmatrix},$$

$$M_{21} = \begin{pmatrix} \frac{-1+x^4}{x^2\left(\pi+3\pi x^2\right)} & -\frac{3+x^2}{2\pi+6\pi x^2} & -\frac{3+x^2}{2\pi+6\pi x^2} & \frac{1}{2\pi x^2} & \frac{1}{2\pi x^2} \\ -\frac{1+14x^2+x^4}{4\pi+8\pi x^2-12\pi x^4} & -\frac{3+11x^2+49x^4+x^6}{8\left(-1+x^2\right)^2\left(\pi+3\pi x^2\right)} & -\frac{3+x^2}{8\left(\pi+3\pi x^2\right)} & \frac{1}{2\pi(-1+x)^2} & \frac{1}{2\pi(1+x)^2} \\ -\frac{1+14x^2+x^4}{4\pi+8\pi x^2-12\pi x^4} & -\frac{3+x^2}{8\left(\pi+3\pi x^2\right)} & -\frac{3+11x^2+49x^4+x^6}{8\left(-1+x^2\right)^2\left(\pi+3\pi x^2\right)} & \frac{1}{2\pi(1+x)^2} & \frac{1}{2\pi(-1+x)^2} \\ -\frac{1+14x^2+x^4}{4\pi x^2+8\pi x^4-12\pi x^6} & -\frac{3+x^2}{2(-1+x)^2\left(\pi+3\pi x^2\right)} & -\frac{3+x^2}{2(1+x)^2\left(\pi+3\pi x^2\right)} & \frac{1+49x^2+11x^4+3x^6}{8x^2\left(-1+x^2\right)^2\left(\pi+3\pi x^2\right)} & \frac{1}{8\pi x^2} \\ -\frac{1+14x^2+x^4}{4\pi x^2+8\pi x^4-12\pi x^6} & -\frac{3+x^2}{2(1+x)^2\left(\pi+3\pi x^2\right)} & -\frac{3+x^2}{2(-1+x)^2\left(\pi+3\pi x^2\right)} & \frac{1}{8\pi x^2} & \frac{1+49x^2+11x^4+3x^6}{8x^2\left(-1+x^2\right)^2\left(\pi+3\pi x^2\right)} \end{pmatrix}$$

The numerical simulation of the eigenvalues on the interval $(-10; 10)$ shows that all of them have zero real part. Thus symmetric configurations of five point singularities are neutrally stable for $x \in (-10; 10)$.

Appendix A

Unit sphere restrictions for the distances

In this appendix we will derive formulas for the volume of triangular pyramid along with the conditions edges in order to have a unit circumradius.

Consider triangular pyramid ABCD (see Fig. A.1). Let

$$AB = a, AC = b, AD = c, BC = d, BD = e, CD = f,$$
$$PO_1 = p_1, PO_2 = p_2, BH = h, \angle O_1PO_2 = \alpha, OA = r,$$

where $PO_1 \perp CD, PO_2 \perp CD$ and $CP = PD$.

First note that we can find all the flat angles from cosine law in each triangle. Also by cosine law for spherical triangle we have

$$\cos(\angle ACB) = \cos(\angle BCD)\cos(\angle ACD) + \sin(\angle BCD)\sin(\angle ACD)\cos(\alpha), \tag{A.0.1}$$

where α is a dihedral angle between planes of triangles $\triangle ACD$ and $\triangle CBD$. So

$$\cos(\alpha) = \frac{\cos(\angle ACB) - \cos(\angle BCD)\cos(\angle ACD)}{\sin(\angle BCD)\sin(\angle ACD)} =$$
$$= \frac{\frac{b^2+d^2-a^2}{2bd} - \frac{d^2+f^2-e^2}{2df}\frac{b^2+f^2-c^2}{2bf}}{\left(1-\left(\frac{d^2+f^2-e^2}{2df}\right)^2\right)^{\frac{1}{2}}\left(1-\left(\frac{b^2+f^2-c^2}{2bf}\right)^2\right)^{\frac{1}{2}}} =$$

$$= 2\frac{f^2(b^2+d^2-a^2)-(d^2+f^2-e^2)(b^2+f^2-c^2)}{(4d^2f^2-(d^2+f^2-e^2)^2)^{\frac{1}{2}}\,(4b^2f^2-(b^2+f^2-c^2)^2)^{\frac{1}{2}}}.$$

Then from $\triangle BCD$ we have

$$BM = BC\sin(\angle BCD) = d\left(1-\left(\frac{d^2+f^2-e^2}{2df}\right)^2\right)^{\frac{1}{2}},$$

And from $\triangle BHM$ we have

$$h = BH = BM\sin(\alpha) = d\left(1-\left(\frac{d^2+f^2-e^2}{2df}\right)^2\right)^{\frac{1}{2}}\times$$
$$\times\left(1-\frac{4\left[f^2(b^2+d^2-a^2)-(d^2+f^2-e^2)(b^2+f^2-c^2)\right]^2}{\left[4d^2f^2-(d^2+f^2-e^2)^2\right]\left[4b^2f^2-(b^2+f^2-c^2)^2\right]}\right)^{\frac{1}{2}}.$$

And now we can find volume of parallelepiped built on vectors CA, CB, CD

$$V_p = 2S_{\triangle ACD}h = 2\frac{1}{4}\left((b^2+c^2+f^2)^2-2(b^4+c^4+f^4)\right)^{\frac{1}{2}}h,$$

Or

$$V_p = \frac{1}{4}[a^2f^2(-a^2+b^2+c^2+d^2+e^2-f^2)+b^2e^2(a^2-b^2+c^2+d^2-e^2+f^2)+$$
$$+c^2d^2(a^2+b^2-c^2-d^2+e^2+f^2)-a^2b^2d^2-a^2c^2e^2-b^2c^2f^2-d^2e^2f^2]^{\frac{1}{2}}.$$

And volume of pyramid $ABCD$ is

$$V(a,b,c,d,e,f) = V_{ABCD} = \frac{1}{6}V_p =$$
$$= \frac{1}{24}[a^2f^2(-a^2+b^2+c^2+d^2+e^2-f^2)+$$

$$+b^2e^2(a^2-b^2+c^2+d^2-e^2+f^2)+$$
$$+c^2d^2(a^2+b^2-c^2-d^2+e^2+f^2)-$$
$$-a^2b^2d^2-a^2c^2e^2-b^2c^2f^2-d^2e^2f^2]^{\frac{1}{2}}.$$

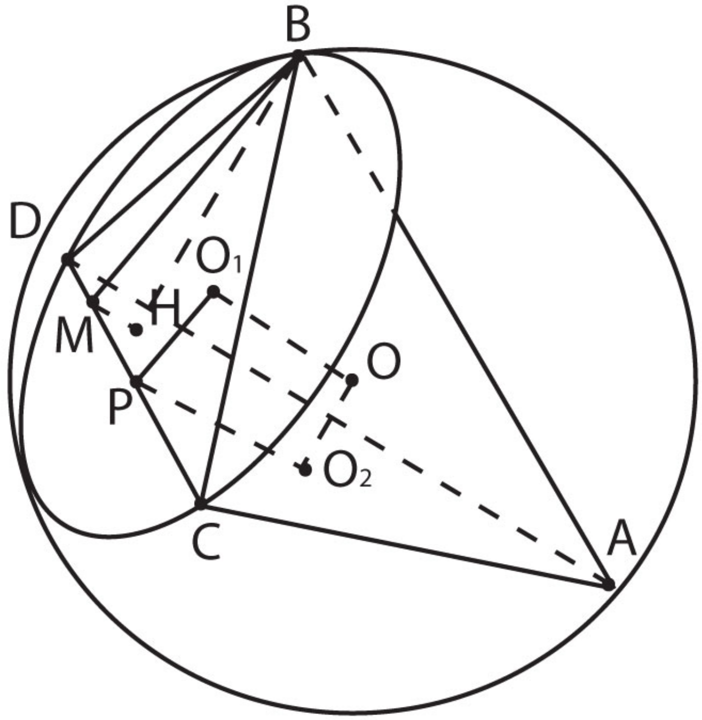


Figure A.1: Pyramid inscribed in a sphere

Now to restrict the radius of the pyramid, we notice that $V_{ABCD} = V_{OABC} + V_{OABD} + V_{OACD} + V_{OBCD}$. By setting $OA = OB = OC = OD = r = 1$ we get

$$V_{OABC} = V(a,b,d) = \frac{1}{24}\left[2(a^2b^2+b^2d^2+d^2a^2)-a^4-b^4-d^4-a^2b^2d^2\right]^{\frac{1}{2}} \quad \text{(A.0.2)}$$
$$V_{OABD} = V(a,c,e) = \frac{1}{24}\left[2(a^2c^2+c^2e^2+e^2a^2)-a^4-c^4-e^4-a^2c^2e^2\right]^{\frac{1}{2}} \quad \text{(A.0.3)}$$
$$V_{OACD} = V(b,c,f) = \frac{1}{24}\left[2(b^2c^2+c^2f^2+f^2b^2)-b^4-c^4-f^4-b^2c^2f^2\right]^{\frac{1}{2}} \quad \text{(A.0.4)}$$

$$V_{OBCD} = V(d,e,f) = \frac{1}{24}\left[2(d^2e^2+e^2f^2+f^2d^2) - d^4 - e^4 - f^4 - d^2e^2f^2\right]^{\frac{1}{2}} \quad \text{(A.0.5)}$$

And thus the restrictions on the sides are

$$\begin{aligned}
&\big[a^2f^2(-a^2+b^2+c^2+d^2+e^2-f^2)+b^2e^2(a^2-b^2+c^2+d^2-e^2+f^2)+\\
&+c^2d^2(a^2+b^2-c^2-d^2+e^2+f^2)-a^2b^2d^2-a^2c^2e^2-b^2c^2f^2-d^2e^2f^2\big]^{\frac{1}{2}} =\\
&= \left[2(a^2b^2+b^2d^2+d^2a^2)-a^4-b^4-d^4-a^2b^2d^2\right]^{\frac{1}{2}}+\\
&+\left[2(a^2c^2+c^2e^2+e^2a^2)-a^4-c^4-e^4-a^2c^2e^2\right]^{\frac{1}{2}}+\\
&+\left[2(b^2c^2+c^2f^2+f^2b^2)-b^4-c^4-f^4-b^2c^2f^2\right]^{\frac{1}{2}}+\\
&+\left[2(d^2e^2+e^2f^2+f^2d^2)-d^4-e^4-f^4-d^2e^2f^2\right]^{\frac{1}{2}}.
\end{aligned} \quad \text{(A.0.6)}$$

Another useful relation we can get from the picture is value of a in terms of b, c, d, e, f, r. From the right triangles $\triangle OO_1P$ and $\triangle OO_2P$ we have

$$\cos\angle OPO_1 = \frac{O_1P}{OP},$$
$$\cos\angle OPO_1 = \frac{O_2P}{OP}.$$

Also, recall that $\angle OPO_1 + \angle OPO_2 = \alpha$. From right triangles $\triangle O_1PC$ and $\triangle O_2PC$ we have

$$O_1P^2 = O_1C^2 - \left(\frac{CD}{2}\right)^2 = \frac{d^2e^2f^2}{(d+e+f)(-d+e+f)(d-e+f)(d+e-f)} - \frac{f^2}{4},$$
$$O_2P^2 = O_2C^2 - \left(\frac{CD}{2}\right)^2 = \frac{b^2c^2f^2}{(b+c+f)(-b+c+f)(b-c+f)(b+c-f)} - \frac{f^2}{4}.$$

From right triangle $\triangle OPC$ we have

$$OP^2 = r^2 - \frac{f^2}{4}$$

Now, since

$$\cos\alpha = \cos\angle OPO_1 \cos\angle OPO_2 - \sin\angle OPO_1 \sin\angle OPO_2,$$

from (A.0.1) we have

$$\begin{aligned}\cos\angle ACB = \frac{(b^2-c^2+f^2)(d^2-e^2+f^2)}{4bdf^2} + \\ +\sqrt{1-\frac{(b^2-c^2+f^2)^2}{4b^2f^2}}\sqrt{1-\frac{(d^2-e^2+f^2)^2}{4d^2f^2}} \times \\ \left(\sqrt{\frac{f^2(b^2+c^2-f^2)^2}{\left(b^4+(c^2-f^2)^2-2b^2(c^2+f^2)\right)(f^2-4r^2)}}\times\right. \\ \sqrt{\frac{f^2(d^2+e^2-f^2)^2}{\left(d^4+(e^2-f^2)^2-2d^2(e^2+f^2)\right)(f^2-4r^2)}} - \\ -\sqrt{1-\frac{f^2(b^2+c^2-f^2)^2}{\left(b^4+(c^2-f^2)^2-2b^2(c^2+f^2)\right)(f^2-4r^2)}}\times \\ \left.\sqrt{1-\frac{f^2(d^2+e^2-f^2)^2}{\left(d^4+(e^2-f^2)^2-2d^2(e^2+f^2)\right)(f^2-4r^2)}}\right),\end{aligned}$$

and from triangle $\triangle ABC$

$$a^2 = b^2 + c^2 - 2bc\cos\angle ACB, \Rightarrow$$

$$a^2 = b^2 + c^2 - 2bc\left(\frac{(b^2-c^2+f^2)(d^2-e^2+f^2)}{4bdf^2} + \right.$$

$$
\begin{aligned}
&+\sqrt{1-\frac{\left(b^2-c^2+f^2\right)^2}{4b^2f^2}}\sqrt{1-\frac{\left(d^2-e^2+f^2\right)^2}{4d^2f^2}}\times \\
&\left(\sqrt{\frac{f^2\left(b^2+c^2-f^2\right)^2}{\left(b^4+\left(c^2-f^2\right)^2-2b^2\left(c^2+f^2\right)\right)\left(f^2-4r^2\right)}}\times\right. \\
&\sqrt{\frac{f^2\left(d^2+e^2-f^2\right)^2}{\left(d^4+\left(e^2-f^2\right)^2-2d^2\left(e^2+f^2\right)\right)\left(f^2-4r^2\right)}}- \\
&-\sqrt{1-\frac{f^2\left(b^2+c^2-f^2\right)^2}{\left(b^4+\left(c^2-f^2\right)^2-2b^2\left(c^2+f^2\right)\right)\left(f^2-4r^2\right)}}\times \\
&\left.\left.\sqrt{1-\frac{f^2\left(d^2+e^2-f^2\right)^2}{\left(d^4+\left(e^2-f^2\right)^2-2d^2\left(e^2+f^2\right)\right)\left(f^2-4r^2\right)}}\right)\right).
\end{aligned}
$$

Appendix B

Components of second variation for cube and icosahedron

Matrix $\mathbf{B}$ can be written as

$$\mathbf{B} = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 & B_5 \\ B_2 & B_6 & B_7 & B_8 & B_9 \\ B_3 & B_7 & B_{10} & B_{11} & B_{12} \\ B_4 & B_8 & B_{11} & B_{13} & B_{14} \\ B_5 & B_9 & B_{12} & B_{14} & B_{15} \end{pmatrix},$$

where $B_i, i = 1, .., 15$ are 2×2 matrices

$$B_1 = \begin{pmatrix} -\frac{27}{16}\left(\Gamma^2-\Gamma_\alpha^2\right)\left(6\Gamma^2-\Gamma_\alpha\left(\Gamma_\alpha+3\Gamma_\beta\right)\right) & 0 \\ 0 & -\frac{4}{3}\left(9\Gamma^4-10\Gamma_\alpha^2\Gamma^2+\Gamma_\alpha^4\right) \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \frac{27}{128}\sqrt{3}\left(\Gamma^2-\Gamma_\alpha^2\right)\left(15\Gamma+5\Gamma_\alpha+6\Gamma_\beta\right) & -\frac{27}{128}\left(\Gamma^2-\Gamma_\alpha^2\right)\left(15\Gamma+5\Gamma_\alpha+6\Gamma_\beta\right) \\ -\frac{1}{16}\left(\Gamma-\Gamma_\alpha\right)\left(\Gamma+\Gamma_\alpha\right)\left(45\Gamma+29\Gamma_\alpha\right) & -\frac{1}{16}\sqrt{3}\left(\Gamma-\Gamma_\alpha\right)\left(\Gamma+\Gamma_\alpha\right)\left(45\Gamma+29\Gamma_\alpha\right) \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & -\frac{27}{64}\left(\Gamma^2-\Gamma_\alpha^2\right)\left(15\Gamma-5\Gamma_\alpha-6\Gamma_\beta\right) \\ -\frac{1}{8}\left(45\Gamma-29\Gamma_\alpha\right)\left(\Gamma-\Gamma_\alpha\right)\left(\Gamma+\Gamma_\alpha\right) & 0 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} \frac{9}{32}\sqrt{\frac{3}{2}}\left(\Gamma^2-\Gamma_\alpha^2\right)\left(\Gamma+\Gamma_\beta\right)\left(33\Gamma+13\Gamma_\alpha+12\Gamma_\beta\right) & -\frac{9\left(\Gamma^2-\Gamma_\alpha^2\right)\left(\Gamma-\Gamma_\beta\right)\left(15\Gamma-5\Gamma_\alpha+12\Gamma_\beta\right)}{32\sqrt{2}} \\ -\frac{21\left(\Gamma-\Gamma_\alpha\right)\left(\Gamma+\Gamma_\alpha\right)^2\left(\Gamma+\Gamma_\beta\right)}{4\sqrt{2}} & -\frac{15}{4}\sqrt{\frac{3}{2}}\left(\Gamma-\Gamma_\alpha\right)\left(\Gamma+\Gamma_\alpha\right)^2\left(\Gamma-\Gamma_\beta\right) \end{pmatrix},$$

$$B_5 = \begin{pmatrix} -\frac{81}{16}\sqrt{\frac{3}{2}}\left(\Gamma^2-\Gamma_\alpha^2\right)\left(\Gamma^2-\Gamma_\beta^2\right) & \frac{81\left(\Gamma^2-\Gamma_\alpha^2\right)\left(\Gamma^2-\Gamma_\beta^2\right)}{16\sqrt{2}} \\ \frac{3\left(\Gamma^2-\Gamma_\alpha^2\right)\left(\Gamma^2-\Gamma_\beta^2\right)}{2\sqrt{2}} & \frac{3}{2}\sqrt{\frac{3}{2}}\left(\Gamma^2-\Gamma_\alpha^2\right)\left(\Gamma^2-\Gamma_\beta^2\right) \end{pmatrix},$$

$$B_6 = \begin{pmatrix} -\frac{3}{32}(\Gamma+\Gamma_\alpha)(58\Gamma+5\Gamma_\alpha+27\Gamma_\beta) & 0 \\ 0 & -\frac{3}{32}(\Gamma+\Gamma_\alpha)(58\Gamma+5\Gamma_\alpha+27\Gamma_\beta) \end{pmatrix},$$
$$B_7 = \begin{pmatrix} -\frac{3}{2}(\Gamma^2-\Gamma_\alpha^2) & \frac{3}{2}\sqrt{3}(\Gamma^2-\Gamma_\alpha^2) \\ \frac{3}{2}\sqrt{3}(\Gamma_\alpha^2-\Gamma^2) & -\frac{3}{2}(\Gamma^2-\Gamma_\alpha^2) \end{pmatrix},$$
$$B_8 = \begin{pmatrix} -\frac{9(\Gamma+\Gamma_\alpha)(\Gamma+\Gamma_\beta)(15\Gamma+5\Gamma_\alpha+6\Gamma_\beta)}{8\sqrt{2}} & 0 \\ 0 & -\frac{9(\Gamma+\Gamma_\alpha)(\Gamma-\Gamma_\beta)(5\Gamma+3\Gamma_\beta)}{4\sqrt{2}} \end{pmatrix},$$
$$B_9 = \begin{pmatrix} \frac{45(\Gamma+\Gamma_\alpha)\left(\Gamma^2-\Gamma_\beta^2\right)}{8\sqrt{2}} & 0 \\ 0 & \frac{45(\Gamma+\Gamma_\alpha)\left(\Gamma^2-\Gamma_\beta^2\right)}{8\sqrt{2}} \end{pmatrix},$$
$$B_{10} = \begin{pmatrix} -\frac{3}{32}(\Gamma-\Gamma_\alpha)(58\Gamma-5\Gamma_\alpha-27\Gamma_\beta) & 0 \\ 0 & -\frac{3}{32}(\Gamma-\Gamma_\alpha)(58\Gamma-5\Gamma_\alpha-27\Gamma_\beta) \end{pmatrix},$$
$$B_{11} = \begin{pmatrix} -\frac{45\left(\Gamma^2-\Gamma_\alpha^2\right)(\Gamma+\Gamma_\beta)}{16\sqrt{2}} & 0 \\ \frac{45}{16}\sqrt{\frac{3}{2}}\left(\Gamma^2-\Gamma_\alpha^2\right)(\Gamma+\Gamma_\beta) & 0 \end{pmatrix},$$
$$B_{12} = \begin{pmatrix} \frac{45(\Gamma-\Gamma_\alpha)\left(\Gamma^2-\Gamma_\beta^2\right)}{16\sqrt{2}} & \frac{45}{16}\sqrt{\frac{3}{2}}(\Gamma-\Gamma_\alpha)\left(\Gamma^2-\Gamma_\beta^2\right) \\ -\frac{45}{16}\sqrt{\frac{3}{2}}(\Gamma-\Gamma_\alpha)\left(\Gamma^2-\Gamma_\beta^2\right) & \frac{45(\Gamma-\Gamma_\alpha)\left(\Gamma^2-\Gamma_\beta^2\right)}{16\sqrt{2}} \end{pmatrix},$$
$$B_{13} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$
$$B_{14} = \begin{pmatrix} \frac{1}{4}(\Gamma+\Gamma_\alpha)(39\Gamma+9\Gamma_\alpha+32\Gamma_\beta)\left(\Gamma^2-\Gamma_\beta^2\right) & 0 \\ 0 & \frac{1}{4}(\Gamma+\Gamma_\alpha)(15\Gamma+9\Gamma_\alpha-22\Gamma_\beta)\left(\Gamma^2-\Gamma_\beta^2\right) \end{pmatrix},$$
$$B_{15} = \begin{pmatrix} -\frac{1}{2}\left(12\Gamma^2-5\Gamma_\beta^2-9\Gamma_\alpha\Gamma_\beta\right)\left(\Gamma^2-\Gamma_\beta^2\right) & 0 \\ 0 & -\frac{1}{2}\left(12\Gamma^2-5\Gamma_\beta^2-9\Gamma_\alpha\Gamma_\beta\right)\left(\Gamma^2-\Gamma_\beta^2\right) \end{pmatrix},$$
$$a = -\frac{1}{4}(\Gamma+\Gamma_\alpha)(\Gamma+\Gamma_\beta)\left(116\Gamma^2+145\Gamma_\beta\Gamma+9\Gamma_\alpha^2+36\Gamma_\beta^2+\Gamma_\alpha(65\Gamma+49\Gamma_\beta)\right),$$
$$b = -\frac{1}{4}(\Gamma+\Gamma_\alpha)(\Gamma-\Gamma_\beta)\left(62\Gamma^2-19\Gamma_\beta\Gamma-9\Gamma_\alpha^2-36\Gamma_\beta^2+\Gamma_\alpha(5\Gamma_\beta-7\Gamma)\right).$$

Matrix $\mathbf{C}$ can be written as

$$\mathbf{C} = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 \\ C_2 & C_5 & C_6 & C_7 \\ C_3 & C_6 & C_8 & C_9 \\ C_4 & C_8 & C_9 & C_{10} \end{pmatrix},$$

where $C_i, i = 1, .., 10$ are 2×2 matrices

$$C_1 = \begin{pmatrix} -\frac{1}{8}(\Gamma+\Gamma_\alpha)\left(203\Gamma+37\Gamma_\alpha+25\sqrt{5}\Gamma_\beta\right) & 0 \\ 0 & -\frac{1}{8}(\Gamma+\Gamma_\alpha)\left(203\Gamma+37\Gamma_\alpha+25\sqrt{5}\Gamma_\beta\right) \end{pmatrix},$$

$$C_2 = \begin{pmatrix} \frac{27}{128}\left(-1+\sqrt{5}\right)(\Gamma^2-\Gamma_\alpha^2) & \frac{1}{128}\sqrt{2643290+922258\sqrt{5}}\,(\Gamma_\alpha^2-\Gamma^2) \\ \frac{1}{128}\sqrt{1659290-61742\sqrt{5}}\,(\Gamma_\alpha^2-\Gamma^2) & \frac{27}{128}\left(-1+\sqrt{5}\right)(\Gamma^2-\Gamma_\alpha^2) \end{pmatrix},$$

$$C_3 = \begin{pmatrix} \frac{1}{8}(\Gamma+\Gamma_\alpha)\left(53\Gamma+187\Gamma_\alpha-25\sqrt{5}\Gamma_\beta\right) & 0 \\ 0 & -\frac{1}{8}(\Gamma+\Gamma_\alpha)\left(53\Gamma+187\Gamma_\alpha-25\sqrt{5}\Gamma_\beta\right) \end{pmatrix},$$

$$C_4 = \begin{pmatrix} \frac{1}{128}\left(877-77\sqrt{5}\right)(\Gamma^2-\Gamma_\alpha^2) & \frac{27}{64}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}\,(\Gamma^2-\Gamma_\alpha^2) \\ \frac{27}{64}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}\,(\Gamma^2-\Gamma_\alpha^2) & \frac{1}{128}\left(1123-323\sqrt{5}\right)(\Gamma^2-\Gamma_\alpha^2) \end{pmatrix},$$

$$C_5 = \begin{pmatrix} -\frac{1}{8}(\Gamma-\Gamma_\alpha)\left(203\Gamma-37\Gamma_\alpha-25\sqrt{5}\Gamma_\beta\right) & 0 \\ 0 & -\frac{1}{8}(\Gamma-\Gamma_\alpha)\left(203\Gamma-37\Gamma_\alpha-25\sqrt{5}\Gamma_\beta\right) \end{pmatrix},$$

$$C_6 = \begin{pmatrix} \frac{1}{128}\left(-1123+323\sqrt{5}\right)(\Gamma^2-\Gamma_\alpha^2) & -\frac{27}{64}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}\,(\Gamma^2-\Gamma_\alpha^2) \\ -\frac{27}{64}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}\,(\Gamma^2-\Gamma_\alpha^2) & \frac{1}{128}\left(-877+77\sqrt{5}\right)(\Gamma^2-\Gamma_\alpha^2) \end{pmatrix},$$

$$C_7 = \begin{pmatrix} \frac{1}{8}(\Gamma-\Gamma_\alpha)\left(53\Gamma-187\Gamma_\alpha+25\sqrt{5}\Gamma_\beta\right) & 0 \\ 0 & -\frac{1}{8}(\Gamma-\Gamma_\alpha)\left(53\Gamma-187\Gamma_\alpha+25\sqrt{5}\Gamma_\beta\right) \end{pmatrix},$$

$$C_8 = \begin{pmatrix} -\frac{1}{8}(\Gamma+\Gamma_\alpha)\left(203\Gamma+37\Gamma_\alpha+25\sqrt{5}\Gamma_\beta\right) & 0 \\ 0 & -\frac{1}{8}(\Gamma+\Gamma_\alpha)\left(203\Gamma+37\Gamma_\alpha+25\sqrt{5}\Gamma_\beta\right) \end{pmatrix},$$

$$C_9 = \begin{pmatrix} \frac{27}{128}\left(-1+\sqrt{5}\right)(\Gamma^2-\Gamma_\alpha^2) & \frac{1}{128}\sqrt{1659290-61742\sqrt{5}}\,(\Gamma_\alpha^2-\Gamma^2) \\ \frac{1}{128}\sqrt{2643290+922258\sqrt{5}}\,(\Gamma_\alpha^2-\Gamma^2) & \frac{27}{128}\left(-1+\sqrt{5}\right)(\Gamma^2-\Gamma_\alpha^2) \end{pmatrix},$$

$$C_{10} = \begin{pmatrix} -\frac{1}{8}(\Gamma-\Gamma_\alpha)\left(203\Gamma-37\Gamma_\alpha-25\sqrt{5}\Gamma_\beta\right) & 0 \\ 0 & -\frac{1}{8}(\Gamma-\Gamma_\alpha)\left(203\Gamma-37\Gamma_\alpha-25\sqrt{5}\Gamma_\beta\right) \end{pmatrix}.$$

Matrix $\mathbf{D}$ can be written as

$$\mathbf{D} = \begin{pmatrix} D_1 & D_2 & D_3 & D_4 & D_5 \\ D_2 & D_6 & D_7 & D_8 & D_9 \\ D_3 & D_7 & D_{10} & D_{11} & D_{12} \\ D_4 & D_8 & D_{11} & D_{13} & D_{14} \\ D_5 & D_9 & D_{12} & D_{14} & D_{15} \end{pmatrix},$$

where $D_i, i = 1, .., 15$ are 2×2 matrices

$$D_1 = \begin{pmatrix} -\frac{5}{8}(\Gamma+\Gamma_\alpha)\left(33\Gamma+2\Gamma_\alpha+5\sqrt{5}\Gamma_\beta\right) & 0 \\ 0 & -\frac{5}{8}(\Gamma+\Gamma_\alpha)\left(33\Gamma+2\Gamma_\alpha+5\sqrt{5}\Gamma_\beta\right) \end{pmatrix},$$

$$D_2 = \begin{pmatrix} -\frac{5}{16}\left(1+\sqrt{5}\right)(\Gamma^2-\Gamma_\alpha^2) & \frac{5(\Gamma^2-\Gamma_\alpha^2)}{4\sqrt{2+\frac{2}{\sqrt{5}}}} \\ \frac{5(\Gamma_\alpha^2-\Gamma^2)}{4\sqrt{2+\frac{2}{\sqrt{5}}}} & -\frac{5}{16}\left(1+\sqrt{5}\right)(\Gamma^2-\Gamma_\alpha^2) \end{pmatrix},$$

$$D_3 = \begin{pmatrix} 0 & -\frac{25(\Gamma^2-\Gamma_\alpha^2)(5((5+\sqrt{5})\Gamma+(1+\sqrt{5})\Gamma_\beta)-(5+\sqrt{5})\Gamma_\alpha)}{2(5+\sqrt{5})} \\ -\frac{25(\Gamma^2-\Gamma_\alpha^2)(5((5+\sqrt{5})\Gamma+(1+\sqrt{5})\Gamma_\beta)-(5+\sqrt{5})\Gamma_\alpha)}{2(5+\sqrt{5})} & 0 \end{pmatrix},$$

$$D_4 = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix},$$

$$c = -\frac{5}{8}(\Gamma+\Gamma_\alpha)(\Gamma+\Gamma_\beta)\left(3\left(9+\sqrt{5}\right)\Gamma+3\left(1+\sqrt{5}\right)\Gamma_\alpha+4\sqrt{5}\Gamma_\beta\right),$$

$$d = \frac{5(\Gamma+\Gamma_\alpha)(\Gamma-\Gamma_\beta)\left(\left(33+9\sqrt{5}\right)\Gamma-3\left(1+\sqrt{5}\right)\Gamma_\alpha+2\left(5+3\sqrt{5}\right)\Gamma_\beta\right)}{4\left(3+\sqrt{5}\right)},$$

$$D_5 = \begin{pmatrix} \frac{15}{4}\sqrt{5}(\Gamma+\Gamma_\alpha)\left(\Gamma^2-\Gamma_\beta^2\right) & 0 \\ 0 & \frac{15}{4}\sqrt{5}(\Gamma+\Gamma_\alpha)\left(\Gamma^2-\Gamma_\beta^2\right) \end{pmatrix},$$

$$D_6 = \begin{pmatrix} -\frac{5}{8}(\Gamma-\Gamma_\alpha)\left(33\Gamma-2\Gamma_\alpha-5\sqrt{5}\Gamma_\beta\right) & 0 \\ 0 & -\frac{5}{8}(\Gamma-\Gamma_\alpha)\left(33\Gamma-2\Gamma_\alpha-5\sqrt{5}\Gamma_\beta\right) \end{pmatrix},$$

$$D_7 = \begin{pmatrix} e & f \\ f & e \end{pmatrix},$$

$$e = -\frac{25}{8}\sqrt{10-2\sqrt{5}}(\Gamma^2-\Gamma_\alpha^2)\left(5\Gamma+\Gamma_\alpha-\sqrt{5}\Gamma_\beta\right),$$

$$f = -\frac{25(\Gamma^2-\Gamma_\alpha^2)\left(5\left(5+3\sqrt{5}\right)\Gamma+\left(5+3\sqrt{5}\right)\Gamma_\alpha-5\left(3+\sqrt{5}\right)\Gamma_\beta\right)}{4\left(5+\sqrt{5}\right)},$$

$$D_8 = \begin{pmatrix} \frac{5}{8}\left(-2+\sqrt{5}\right)\left(\Gamma^2-\Gamma_\alpha^2\right)\left(\Gamma+\Gamma_\beta\right) & -\frac{5}{16}\sqrt{130+58\sqrt{5}}\left(\Gamma^2-\Gamma_\alpha^2\right)\left(\Gamma-\Gamma_\beta\right) \\ -\frac{5}{8}\sqrt{85-38\sqrt{5}}\left(\Gamma^2-\Gamma_\alpha^2\right)\left(\Gamma+\Gamma_\beta\right) & -\frac{5}{16}\left(11+5\sqrt{5}\right)\left(\Gamma^2-\Gamma_\alpha^2\right)\left(\Gamma-\Gamma_\beta\right) \end{pmatrix},$$

$$D_9 = \begin{pmatrix} \frac{15}{16}\left(5+\sqrt{5}\right)\left(\Gamma-\Gamma_\alpha\right)\left(\Gamma^2-\Gamma_\beta^2\right) & \frac{15}{16}\sqrt{50-10\sqrt{5}}\left(\Gamma-\Gamma_\alpha\right)\left(\Gamma^2-\Gamma_\beta^2\right) \\ -\frac{75\left(\Gamma-\Gamma_\alpha\right)\left(\Gamma^2-\Gamma_\beta^2\right)}{4\sqrt{2\left(5+\sqrt{5}\right)}} & \frac{15}{16}\left(5+\sqrt{5}\right)\left(\Gamma-\Gamma_\alpha\right)\left(\Gamma^2-\Gamma_\beta^2\right) \end{pmatrix},$$

$$D_{10} = \begin{pmatrix} -25\left(\Gamma^2-\Gamma_\alpha^2\right)\left(15\Gamma^2+9\Gamma_\alpha^2-4\sqrt{5}\Gamma_\alpha\Gamma_\beta\right) & 0 \\ 0 & -25\left(\Gamma^2-\Gamma_\alpha^2\right)\left(15\Gamma^2+9\Gamma_\alpha^2-4\sqrt{5}\Gamma_\alpha\Gamma_\beta\right) \end{pmatrix},$$

$$D_{11} = \begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix},$$

$$g = \frac{10\left(\Gamma^2-\Gamma_\alpha^2\right)\left(\Gamma-\Gamma_\beta\right)\left(5\left(-5+2\sqrt{5}\right)\Gamma+\left(5+4\sqrt{5}\right)\Gamma_\alpha-5\left(-1+\sqrt{5}\right)\Gamma_\beta\right)}{-5+\sqrt{5}},$$

$$h = \frac{5\left(\Gamma^2-\Gamma_\alpha^2\right)\left(\Gamma+\Gamma_\beta\right)\left(\left(-35+17\sqrt{5}\right)\Gamma_\alpha+5\left(\left(5+\sqrt{5}\right)\Gamma+2\left(-1+\sqrt{5}\right)\Gamma_\beta\right)\right)}{-5+\sqrt{5}},$$

$$D_{12} = \begin{pmatrix} 0 & 25\sqrt{5}\left(\Gamma^2-\Gamma_\alpha^2\right)\left(\Gamma^2-\Gamma_\beta^2\right) \\ 25\sqrt{5}\left(\Gamma^2-\Gamma_\alpha^2\right)\left(\Gamma^2-\Gamma_\beta^2\right) & 0 \end{pmatrix},$$

$$D_{13} = \begin{pmatrix} k & 0 \\ 0 & l \end{pmatrix},$$

$$k = -\frac{1}{2}\left(\Gamma+\Gamma_\alpha\right)\left(\Gamma+\Gamma_\beta\right)\left(\left(38+5\sqrt{5}\right)\Gamma^2+\right.$$
$$\left.+\left(31+9\sqrt{5}\right)\Gamma_\beta\Gamma+5\sqrt{5}\Gamma_\alpha^2+4\sqrt{5}\Gamma_\beta^2+\left(7+5\sqrt{5}\right)\Gamma_\alpha\left(2\Gamma+\Gamma_\beta\right)\right),$$

$$l = \frac{1}{4\left(3+\sqrt{5}\right)}\left(\Gamma+\Gamma_\alpha\right)\left(\Gamma-\Gamma_\beta\right)\left(-2\left(89+23\sqrt{5}\right)\Gamma^2+10\left(5+3\sqrt{5}\right)\Gamma_\alpha^2+\right.$$
$$\left.+8\left(1+2\sqrt{5}\right)\Gamma_\alpha\left(2\Gamma-\Gamma_\beta\right)+8\Gamma_\beta\left(\left(12+\sqrt{5}\right)\Gamma+\left(5+3\sqrt{5}\right)\Gamma_\beta\right)\right),$$

$$D_{14} = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix},$$

$$m = -\frac{\left(\Gamma+\Gamma_\alpha\right)\left(5\left(1+\sqrt{5}\right)\Gamma+5\left(-5+3\sqrt{5}\right)\Gamma_\alpha+4\left(-4+3\sqrt{5}\right)\Gamma_\beta\right)\left(\Gamma^2-\Gamma_\beta^2\right)}{2\left(-3+\sqrt{5}\right)},$$

$$n = \frac{\left(\Gamma+\Gamma_\alpha\right)\left(5\left(-11+5\sqrt{5}\right)\Gamma+5\left(-5+3\sqrt{5}\right)\Gamma_\alpha+2\left(17-9\sqrt{5}\right)\Gamma_\beta\right)\left(\Gamma^2-\Gamma_\beta^2\right)}{2\left(-3+\sqrt{5}\right)},$$

$$D_{15} = \begin{pmatrix} -\left(10\Gamma^2-3\Gamma_\beta^2-5\sqrt{5}\Gamma_\alpha\Gamma_\beta\right)\left(\Gamma^2-\Gamma_\beta^2\right) & 0 \\ 0 & -\left(10\Gamma^2-3\Gamma_\beta^2-5\sqrt{5}\Gamma_\alpha\Gamma_\beta\right)\left(\Gamma^2-\Gamma_\beta^2\right) \end{pmatrix}.$$

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